Algebra Final Project

Wolf Allred, Kenzie McLean, Maxwell Plummer

April 2021

Introduction

Topology is the study of the properties of spaces independent of continuous deformations. This is a rich field of study which often makes use of tools from other branches of mathematics. Of particular note for this project are the fields of algebraic topology and category theory. Algebraic topology is the intersection of its two eponymous fields, and it is useful for developing methods to perform computations which would otherwise be impossible. One important concept in algebraic topology is the computation of fundamental groups. This includes finding the fundamental group of an arbitrary space, and finding a space with an arbitrary fundamental group. Category theory, on the other hand, is a field which has grown out of topology and which studies the basic fundamental structures of mathematics. It is useful for generalizing concepts between different branches of mathematics, and translating between them.

The goal of this paper is to demonstrate that for every finitely presented group G, there is some space X which has fundamental group G. To show this, we will give some background in topology, free groups, and basic category theory. With these tools developed, we will present and make use of Van Kampen's theorem, a powerful method of computing fundamental groups. Together, these will give us the necessary tools to construct a space with a fundamental group isomorphic to an arbitrary finitely presented group.

Topology

We do not assume any background in Topology, so we will quickly give the necessary definitions.

Basic Definitions

Definition. A topological space is a set X equipped with a collection $\mathcal{T}$ of subsets of X, called the topology, obeying the following properties:

- $\varnothing, X \in \mathcal{T}$,
- $\mathcal{T}$ is closed under arbitrary unions,
- $\mathcal{T}$ is closed under finite intersections.

If a set $U \subseteq X$ is in $\mathcal{T}$ we call it *open.*

Definition. A map $f : X \to Y$ between topological spaces is called *continuous* if $f^{-1}(V)$ is open in X for every open set V of Y.

Remark. A common example of a topological space is a metric space, a set equipped with a distance function. In a metric space, the open sets are unions of open balls.

Definition. A topological space X is path-connected if, for every $x, y \in X$, there exists a continuous function $f : [0,1] \to X$ with $f(0) = x$ and $f(1) = y$.

The Fundamental Group

The fundamental group is a complex concept which might be the subject of a large portion of a beginning class in algebraic topology. We attempt to present the concept here extremely quickly.

Definition. Let X be a topological space and $f, g : [0,1] \to X$ two paths with $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. f and g are *path homotopic* if there exists a continuous map

$$H : [0,1] \times [0,1] \to X$$

satisfying $H(s,0) = f(s)$, $H(s,1) = g(s)$, $H(0,t) = x_0$, and $H(1,t) = x_1$. H is called a *homotopy* between f and g.

Lemma. *We can define a relation $f \simeq g$ if f is path homotopic to g. This is an equivalence relation. We denote the equivalence class of f under this relation by $[f]$.*

This notion of path homotopies is extremely important in algebraic topology and merits further discussion. As a brief intuition, you can think of the homotopy H as smoothly deforming f into g. To support this intuition, for each $t \in [0,1]$ we can define $f_t(s) = H(s,t)$. For a less rigorous intuition, you can think of this equivalence relation as stating "two paths are the same if I can squiggle one into the other" (while keeping the endpoints fixed).

Definition. If f and g are paths in X with $f(1) = g(0)$, we define the concatenation of f with g by

$$(f * g)(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Intuitively, you can think of this as going first through f with twice the speed, then through g with twice the speed.

Now we have the tools to construct the fundamental group. Let (X, x_0) be a topological space with a specified basepoint x_0. The members of our group are homotopy classes of loops f in X, i.e. functions f with $f(0) = f(1) = x_0$. We define a group operation $*$ by $[f] * [g] = [f * g]$. We call this group the fundamental group, denoted by $\pi_1(X, x_0)$. The identity of this group is the trivial loop, with the constant map to x_0 as a representative. The inverse of $[f]$ is $[\bar{f}]$, where

$$\bar{f}(s) = f(1-s).$$

You can think of this as f being traversed backwards, so $f * \bar{f}$ is path homotopic to a constant loop. Associativity of $\pi_1(X, x_0)$ also holds. Even though it is usually not the case that $(f * g) * h = f * (g * h)$, a homotopy can be constructed between them.

Remark. In a path-connected space X, the fundamental group of X does not depend on the basepoint x_0. In this case, $\pi_1(X, x_0)$ is sometimes written as simply $\pi_1(X)$.

Now we will give some examples of fundamental groups of common spaces. Consider $\mathbb{R}$ with the standard topology with basepoint of 0, and let $f : [0,1] \to \mathbb{R}$ be a path. Then we have a homotopy between f and a constant map, given by

$$H(s,t) = (1-t)f(s).$$

Therefore, $\pi_1(\mathbb{R}, 0)$ is the trivial group (a group with one element). As a more complicated example, consider the circle S^1, represented as the unit circle sitting in the complex plane. While the construction of the fundamental group is somewhat long, we can give a heuristic argument. For each integer n, define a map $f_n : [0,1] \to S^1$ by

$$f_n(s) = e^{2\pi i n s}.$$

so the map f_n loops around the circle n times. It can be shown that every loop f is homotopic to f_n for some n. Further, it can also be shown that if $n \neq m$, then $f_n \not\simeq f_m$.

Attaching Spaces

The final topological idea we will discuss is that of attaching spaces, particularly attaching disks to spaces. We denote the closed disk as D^2. We begin with a space X we would like to attach a disk to, and define an attaching map

$f : S^1 \to X$. Then, we define an equivalence relation $x \sim f(x)$ for $x \in S^1 \subset D^2$. Note that if the map f is not injective, this equivalence relation will identify different points of S^1 together. The space X' is the disjoint union of X with D^2 modulo this equivalence relation.

As an example of this, if we think of S^1 as the unit circle in $\mathbb{R}^2$, we can let f be the inclusion of S^1 into $\mathbb{R}^2$. The resultant space will look like $\mathbb{R}^2$ with a hemisphere protruding from it.

As with many of the concepts we discuss here, we can easily generalize this to higher dimensions. In general, the boundary of the n-dimensional disk D^n is the $n-1$ dimensional sphere S^{n-1}, so we can attach this disk to a space X with a map $S^{n-1} \to X$ in the same way as before. It also makes sense to attach an interval in this way, by defining a map from the two end points to a space.

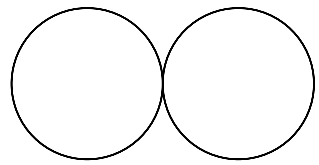


Figure 1: $S^1 \wedge S^1$

Free Groups

Free groups are one area in which algebraic insights can best achieved through topological methods. There are several common questions one might ask about groups, such as what subgroups it has, or how to tell what elements are in a subgroup, which are best answered using the theory of covering spaces and lifts.

To illustrate this, we will construct the free group on 2 variables as the fundamental group of the space $S^1 \wedge S^1$.

Fundamental Group of $S^1 \wedge S^1$

First, we will introduce the concept of wedge sums. Let X and Y be topological spaces with basepoints x and y, respectively. The *wedge sum* $X \wedge Y$ is the disjoint union of X with Y, modulo the relation $x \simeq y$. The main example we'll be dealing with is $S^1 \wedge S^1$, which looks like the numeral 8. This is shown in Figure 1. We can generalize this to define "The Rose of n petals" $R_n = \bigwedge_{j=1}^{n} S^1$, by identifying one point from each of the n circles.

Now we can consider the fundamental group of $S^1 \wedge S^1$. Chose a direction to traverse around the left circle, and label that path a. Do the same for the right circle, and label that path b. We'll label the paths in the opposite directions by $\bar{a}$ and $\bar{b}$, respectively. We'll think of these labels as loops (or homotopy classes of loops) in $S^1 \cap S^1$. If we denote the trivial loop by e we can make the observation that $a\bar{a} \simeq e \simeq \bar{a}a$; and likewise for b.

In our discussion of the fundamental group of S^1 we made the claim that every loop in S^1 was path homotopic to a specific sort of loop corresponding to an integer; we'll make a similar claim here. Let us define a class of loops $w : [0,1] \to S^1 \wedge S^1$ by

$$w = x_1 x_2 \ldots x_k.$$

such that each $x_i \in \{a, b, \bar{a}, \bar{b}\}$, and $x_i \neq \bar{x}_{i+1}$. The loop is defined by first going around x_1, then x_2, et cetera. We claim that every nontrivial loop is of this form. As with the case of S^1, we won't spend time justifying this claim[1]. Now that we have a description of the elements of the group, which we refer to as *reduced words*, we need to give the group an operation that coincides with the concatenation of loops and make sure it satisfies the group axioms. We'll call the identity of this group the "empty word."

[1]If you are a student wanting to learn this topic, the texts by Munkres and Hatcher in the references cover this fairly well. Also, this is the topic of a large portion of Math 5520, the

The operation for this group will also be a concatenation. If $w_1 = x_1 \dots x_k$ and $w_2 = y_1 \dots y_\ell$, then we let $w_1 \cdot w_2 = x_1 \dots, x_k y_1 \dots y_\ell$. There is a potential problem here, namely if $x_k = \bar{y}_1$, then the concatenation is not reduced. The solution is simple, we just remove $x_k y_1$ from our expression and leave it as $x_1 \dots x_{k-1} y_2 \dots y_\ell$. If $x_{k-1} = \bar{y}_2$ we repeat the process, which will terminate in finite steps. After this reduction process we will be left with a "reduced word," making this a well-defined binary operation. The inverse of $w = x_1 \dots x_k$ is $\bar{w} = \bar{x}_k \dots \bar{x}_2 \bar{x}_1$. Associativity is much harder to show using words and this reductive process, so we appeal to the fact that this group is the fundamental group of $S^1 \times S^1$, and that the concatenation operation on loops is associative (up to a path homotopy). This group is the **free group on 2 letters,**[2] and is denoted by F_2.

There's nothing particularly special about the fact that two circles were used here. In general, the fundamental group of the rose with n petals R_n is the free group on n letters F_n. We will prove this fact later in the paper.

More Information on Free Groups

Earlier we alluded to questions of subgroups of free groups and when we can determine their members; we would be remiss to entirely neglect discussion of these things. We have not quite developed the topological methods necessary to derive these results, as they involve "folding" of graphs corresponding to subgroups[3]. The first fact we note is that every (nontrivial) subgroup of a free group is free. For a subgroup generated by a single element w, this is easy to see, since $w^k \neq 1$ for all nonzero integers k.

Free groups have the following universal property. If F is the free group generated by a set S, and G is any group, then any function from S to G induces a unique homomorphism from F to G. This gives some motivation for the name "free," since you are free to send the generators wherever you want and still have a homomorphism. You might think of it as analogous to $\mathbb{Z}$ in rings, as $\mathbb{Z}$ has a unique ring homomorphism to any ring R. This seems much more restrictive than the property of free groups, which is true, but it is also true that ring homomorphisms satisfy much more stringent conditions than group homomorphisms.

Group Presentations and Free Products

One common way to represent certain groups is via a **presentation.** A presentation is a set of generators of a group, together with the relations which they satisfy. We write it in the form $G = \langle S | R \rangle$ where S is the set of generators and

second in the undergraduate topology sequence.

[2]You may also see this referred to as a free group on 2 generators. We use the term letters here because it fits in better with the description of elements as "words."

[3]See Mladen Bestvina's notes on this topic: *Folding graphs and applications, d'après Stallings.* In Fall 2021 he is teaching Math 4800, focusing on Geometric Group Theory, and will discuss this sort of thing.

R is the set of relations. A group is **finitely presented** if both S and R are finite.

It is common practice for the relations to be written as members which are equal to the identity, or to each other, e.g. $\langle x, y | xy = yx \rangle$. Note that if we have a relation written by an equality of non-identity elements, we could represent it by finitely many relations of equalities of identity elements. It will be useful for us to mainly consider relations as elements equal to the identity. We now give some examples of group presentations.

- The free group on n letters is presented by $\langle x_1, \ldots, x_n | \varnothing \rangle$.
- The cyclic group of order n, C_n, is presented by $\langle x | x^n \rangle$.
- The dihedral group D_n is presented by $\langle x, r | x^n, r^2, x^{-1}rxr \rangle$.
- The direct product $\mathbb{Z} \times \mathbb{Z}$ is presented by $\langle x, y | x^{-1}y^{-1}xy \rangle$.
- The Klein 4 Group is presented by $\langle x, y | x^2, y^2, (xy)^2 \rangle$

Note that many common groups are not finitely presented. For instance $(\mathbb{R}, +)$ is not, nor is any group which is not finitely generated. There are groups which are finitely generated but not finitely presented, but examples of such groups are convoluted so they are omitted here.

Now we discuss **free products** of groups. Informally, elements of the free product of G with H can be thought of as formal words where the letters are elements of G and H. In a free group, there was a reduction process anytime a letter was preceded or succeeded by its inverse, here there are reductions given by the relations of elements of G and H.

More formally, let G be presented by $\langle S_G | R_G \rangle$ and H by $\langle S_H | R_H \rangle$. Assume that S_G and S_H are disjoint. Then the free product $G * H$ is presented by $\langle S_G \cup S_H | R_G \cup R_H \rangle$.

There aren't many interesting examples of free products, as it's an operation that just takes two groups and puts them together, assuming no relationship between them. But, there are some interesting examples with free groups.

- Consider $\mathbb{Z} * \mathbb{Z}$. We could present $\mathbb{Z}$ by $\langle a | \varnothing \rangle$ or $\langle b | \varnothing \rangle$. In this way, we could present $\mathbb{Z} * \mathbb{Z}$ by $\langle a, b | \varnothing \rangle$. Thus $\mathbb{Z} * \mathbb{Z} \cong F_2$.
- Let F_m and F_n be free groups on m, n letters, respectively. Write their presentations as $\langle x_1, \ldots, x_m | \varnothing \rangle$ and $\langle y_1, \ldots y_n | \varnothing \rangle$. Then $F_m * F_n$ is presented by $\langle x_1, \ldots, x_m, y_1, \ldots, y_n | \varnothing \rangle$, so $F_m * F_n \cong F_{m+n}$.
- For a non-free group example, take $C_2 * C_2$, where a and b are their respective generators. $C_2 * C_2$ is presented by $\langle a, b | a^2, b^2 \rangle$, and elements are just words in a and b where no letter follows itself. As in free groups, the operation is concatenation followed by the reduction process induced by the relations.

A Very Brief Exposition of Category Theory

Introduction

Before diving headfirst into category theory, we should first establish why we are using it. Category theory got its start in algebraic topology, which wikipedia describes as *"... a branch of mathematics that uses tools from abstract algebra to study topological space"*. It is worth noting that there are several algebraic results that may be proved using tools from topology; the relationship goes both ways. Several examples of this interplay have showed up in our class, such as Riemann surfaces, and the proof that $\mathbb{C}$ is algebraically closed. In fact, we have also proved one of the most fundamental results of category theory, the Yoneda Lemma, for unital commutative rings. The categories that we are primarily concerned with in this paper are Top, the category of topological spaces and continuous maps; Group, the category of groups and homomorphisms; and $\mathsf{B}G$, the category consisting of a single object whose morphisms form the group G with the operation of composition. We will give examples involving several other categories as well, though they will use for the main result we are striving towards.

The utility of category theory is that it generalizes all of mathematics in terms of categories, and in doing so provides a natural way to carry results from one area (or category) of math to another. This is much more powerful than it may appear at first glance; category theory goes far further than just moving structures and theorems from one place to another. By describing mathematical concepts in terms that are divorced from the trappings of the objects involved, category theory standardizes methods of proof and provides a perspective in which seemingly different structures laying in distinct fields of study may be viewed as the same construction. Here, we provide an extremely condensed introduction to category theory to assist us in understanding Van Kampen's theorem. In doing so, we hope to provide an accessible introduction to the remarkable power of categorical thinking. One of the main references for categorical concepts in this paper is Emily Riehl's *Category Theory in Context*[4]; we will follow it closely in our definition of categories.

[4] This is an excellent text for learning some category theory. Its strength lies in the examples that it provides, which are great for building intuition.

Definitions

A **category** is given by

- a collection of **objects** X,Y,Z,...
- and **morphisms** f,g,h, which map between the objects...

where

- Each morphism has a **domain** object, and a **codomain** object. The expression $f : X \to Y$ means that f is a morphism with domain X and codomain Y.
- Each object has an associated **identity morphism** $1_X : X \to X$.
- For any pair of morphisms f, g with the codomain of f equalling the domain of g there exists a specified **composite morphism** fg. For example, morphisms $f : X \to Y$ and $g : Y \to Z$ yield a composite morphism $fg : X \to Z$.

Furthermore, we define the composition of morphisms to be associative and unital, with the identity morphisms acting as two sided identities. More explicitly,

- For any morphism $f : X \to Y$ the composites $1_Y f$ and $f 1_X$ both equal f.
- For any composable triple of morphisms f, g, h the composite morphisms $h(gf)$ and $(hg)f$ are equal and are written hgf.

If objects X and Y have morphisms $f : X \to Y$ and $g : Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$ we call the objects **isomorphic**, and the morphisms f and g are **isomorphisms**. Morphisms with their codomain equalling their domain are called **endomorphisms**, and endomorphisms which are also isomorphisms are called **automorphisms**. Generally we distinguish between isomorphisms and automorphisms since isomorphisms generally imply distinct domains and codomains. The notation $\mathrm{Hom}(x, y)$ represents the set of all morphisms from x to y.

We will write the names of categories in Sans Serif to make sure that it is clear when we are discussing a category. It is tradition to name categories after their objects with the associated morphisms being understood through context. A morphism always specififes its domain and codomain, so really all you need to determine a category is its morphisms. Furthermore, The Yoneda Lemma shows that every object of a category is determined by its morphisms, so it really is morphisms that should take center stage when one thinks about categories. With this in mind, we turn now to some examples of categories which are related to this paper, or which are simply of interest.

1. Set is the category with sets as objects and functions as morphisms. The isomorphisms in this category are bijections.

2. Group has groups as objects and homomorphisms as morphisms, and it is from this context that morphisms take their name. Naturally, isomorphic groups are isomorphic in the categorical sense as well. Contained within Group as a subcategory is Ab, the category of abelian groups.

3. Top consists of topological spaces with continuous functions as morphisms. A counterpart to this is Top_*, which has topological spaces with specified base points as objects, and base point preserving continuous functions as morphisms. In the former, isomorphic objects are homeomorphic, and the same goes for the latter, but with the restriction that the homeomorphisms preserve base points.

4. Ring is defined similarly to Group, with rings as objects and ring homomorphisms as morphisms. It should be noted that in the context of this paper we will only be discussing unital commutative rings as those have been the exclusive subject of study within this class. Hence, in particular we define Ring to be the category of unital commutative rings.

5. Mod_R is the category of modules over a ring R with morphisms being module homomorphisms. This notation sometimes refers specifically to left R modules in a non commutative setting, but that is not the case here. This category is usually written as $\mathsf{Vec}_{\mathbb{F}}$ when R is a field $\mathbb{F}$ and similarly written as Ab when R is $\mathbb{Z}$, as $\mathbb{Z}$-modules and abelian groups are equivalent.

One may notice that each of the above categories consists of objects and morphisms that are merely sets and functions with additional structure, and so appear similar to Set. Generally, these are referred to as **concrete categories**. To illustrate that not all categories are like this, we give a few examples which are quite different. Notice that the morphisms here are not necessarily functions.

1. $\mathsf{B}G$ is a category consisting of a single object with morphisms being the elements of the group G. As there is only a single object within the category, it follows that each group element acts as an endomorphism within the category. $\mathsf{B}G$ is granted a group structure through the operation of composition, where pre composition can be thought of as left multiplication and post composition is right multiplication. The group axioms follow from the category axioms, where the identity morphism is the group identity, and associativity follows from the associativity of composite morphisms. Notice that every morphism is an isomorphism in this category because group elements are required to have inverses. Categories in which every morphism is an isomorphism are called **groupoids**.

2. $(\mathbb{R}, \leq)$ may be considered as a category with real numbers as objects and morphisms $x \to y$ when $x \leq y$. Notice that there are no isomorphisms of distinct domain and codomain within this category.

3. Mat_R where R is a ring is a category with positive integers as objects. The set of morphisms between two integers x and y, $\mathrm{Hom}(x, y)$, are all $m \times n$

matrices with entries in R. This satisfies the morphism axioms through matrix composition, and the fact that identity matrices serve as identity morphisms.

4. Sub_S is a category in which the objects are subsets of the specified set S and the morphisms are inclusions. Again, there are no isomorphisms in this category.

5. $\mathsf{Subg}_{\mathbb{Z}}$ has subgroups of $\mathbb{Z}$ as objects and inclusions as morphisms. $\mathsf{Subg}_{\mathbb{Z}}$ is similar to Sub_S in that its morphisms are inclusions, as we will explore in some examples later.

Functors and Diagrams

The introduction to this section mentioned that category theory can be used to carry structures from one area of math to another, but so far we have kept discussion confined to individual categories. Having been introduced to categories within familiar mathematical contexts, it seems natural to define a category Cat with categories as objects. However, we don't yet know what a morphism between categories looks like. Ideally we want such a thing to "preserve" categorical structure, so to this end we introduce functors.

A **functor** $F : \mathsf{C} \to \mathsf{D}$ with domain category C and codomain category D has the following:

- An object $Fc \in \mathsf{D}$ for every object $c \in \mathsf{C}$.
- A morphism $Ff : Fc \to Fc' \in \mathsf{D}$ for any morphism $f : c \to c' \in \mathsf{C}$ where the domain and codomain of Ff are equal to F applied to the domain and codomain of f respectively.

and we require that functors fulfill the axioms:

- For any composable pair $f, g \in \mathsf{C}$ we have that $Ff \circ Fg = F(f \circ g)$.
- For each $c \in \mathsf{C}$, $F1_c = 1_{Fc}$

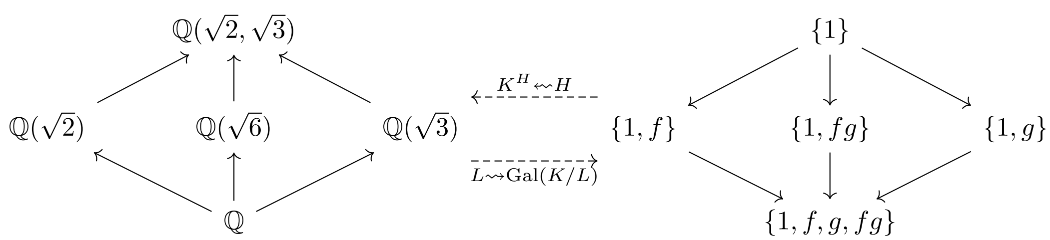
After some consideration, it should be apparent that a functor transfers the structure of the domain category into the target category in much the same way a homomorphism, or a continuous function transfers structure. To get an intuitive understanding of how functors work we will now consider some examples.

- The fundamental group discussed above defines a functor $\pi_1 : \mathsf{Top}_* \to \mathsf{Group}$ taking topological spaces (with a basepoint) to their fundamental groups. Because a continuous function $f : (X, x_0) \to (Y, y_0)$ between spaces induces a homomorphism $f^* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, the functor axioms are satisfied.

- A functor may be endomorphic, mapping a category back onto itself. Such an endofunctor is of the form $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ where $\mathcal{P}(A) \mapsto PA$, which sends a set A to it's power set $PA = \{A' | A' \subset A\}$, and sends a function $f : A \to B$ to $f' : PA \to PB$ that sends subsets A' of A to subsets of $f(A')$ of B.

- The multivariable chain rule may be viewed functorially. Let Euclid_* be the subcategory of Top_* consisting of open subsets of $\mathbb{R}^n$ for any $n \in \mathbb{N}$ with morphisms being differentiable functions. Then, the total derivative evaluated at the basepoint of an open subset serves as a functor $D : \mathsf{Euclid}_* \to \mathsf{Mat}_{\mathbb{R}}$ that sends differentiable functions to their Jacobian matrix at the base point, while the objects, euclidean spaces, are sent to their dimension. More explicitly, given pointed differentiable functions $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^l$ with base points $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, we have that $f \circ g : \mathbb{R}^n \to \mathbb{R}^l$, and we know that $D(gf)$ is the Jacobian at $f \circ g(a)$. As D is a functor, we also know that $D(gf)$ is the composition $D(g)D(f)$ where $D(f)$ and $D(g)$ are the Jacobians of $f(a)$ and $g(b)$. This compositional relationship is exactly the chain rule.

- Let $F \subset K$ be a Galois extension and $\mathsf{Sub}_{K/F}$ be the category where objects are intermediate fields $F \subset L \subset K$ and morphisms are inclusions. Let $G = \mathrm{Gal}(K/F)$. The fundamental theorem of Galois Theory suggests a bijective functor from $\mathsf{Sub}_{K/F}$ to Subg_G. However, after looking at the following diagram for a moment one realizes that this is not a functor as we defined it earlier as all of the arrows are reversed. This is an example of what we call a **contravariant functor**, which is essentially the same as a functor except for that morphisms have their domain and codomain swapped. Explicitely, a contravariant functor $F : \mathsf{C} \to \mathsf{D}$ maps a morphism $f : c \to c'$ in C to a morphism $Ff : c' \to c$ in D. Of course the functor axiom for composition must also be reversed, i.e. for any composable pair $f, g \in \mathsf{C}$, $Ff \circ Fg = F(g \circ f)$.



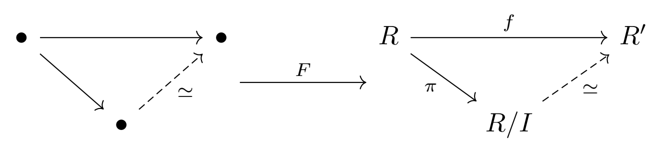
- A functor from a groupoid $\mathsf{B}G$ into a category C defines a G action on the image of the single object in $\mathsf{B}G$[5].

[5]If the codomain of the functor in question is a vector space or a module then this falls into the purview of Representation Theory, which is concerned with studying algebraic structures by representing their elements as linear operators and vector spaces, thereby reducing certain questions down to linear algebra. Because linear algebra is well understood, this can make it easier to look at these abstract structures.

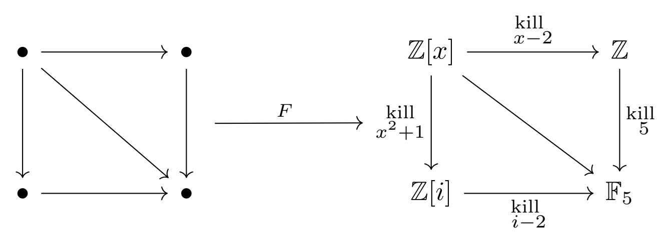
Now it should be apparent how category theory allows one to apply results from areas of math seemingly distinct to each other. All of the examples given so far are quite concrete, but functors are often applied in a more abstract sense. In this vein, we now introduce diagrams, which are central to the study of category theory and essential in our definition of colimits.

Definition. A **diagram** is a functor $F : \mathsf{J} \to \mathsf{C}$ from an **index category** [6] J into a category C.

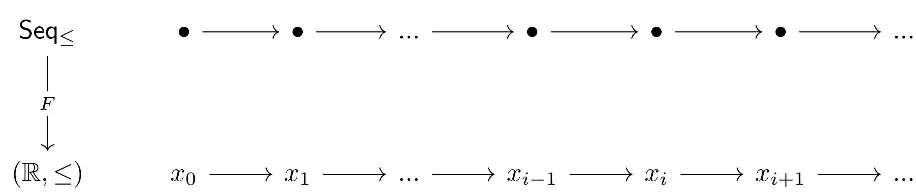
Intuitively, an index category encodes the shape of a diagram of morphisms and objects in the target category. We are already familiar with this concept in the context of **commutative diagrams**, though not every diagram is commutative. For example, The diagram representing the first isomorphism theorem for rings may be viewed functorially



Another example is the commutative diagram from page 338 of *Algebra*(Artin).



A monotonically increasing sequence of real numbers $\{x_i\}_{i=0}^{\infty}$ may be realizing within $(\mathbb{R}, \leq)$ with the diagram $F : \mathsf{Seq}_{\leq} \to (\mathbb{R}, \leq)$



It should be noted that these diagrams alone do not encode the results or proofs of the First isomorphism theorem, or the maps on page 338 of *Algebra*, as it would require a few additional specifications that are outside the scope of this paper.

[6]Formally, this is defined as a **small category**, meaning that it only has a set's worth of arrows (*Category Theory in Context*).

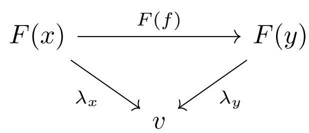
Colimits

Colimits are the main reason for providing the introduction to category theory given in this paper. To understand colimits, we much give one final definition in preparation. We define,

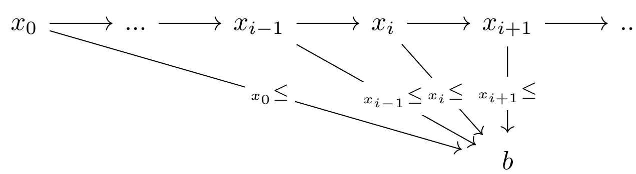
Definition. A **cone under** a diagram $F : \mathsf{J} \to \mathsf{C}$ with a **vertex** $V \in \mathsf{C}$ is a family of morphisms $\text{Cone}(V, \lambda)$ where

$$\text{Cone}(V, \lambda) = \{\lambda_x : F(x) \to v | x \in \mathsf{J}\}$$

The following diagram represents the relationship.

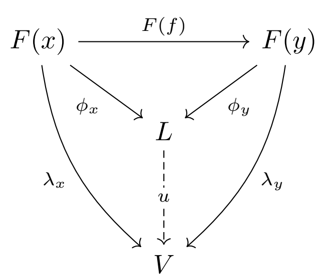


Using the example of the monotonically increasing sequence diagram $F : \mathsf{Seq}_{\leq} \to (\mathbb{R}, \leq)$ from earlier, we see that a cone $(b,_{x_i} \leq)$ under this diagram looks like



The way to interpret this cone is that $b \in \mathbb{R}$ is a number that bounds this whole sequence. Therefore, a diagram of this kind representing an unbounded monotone sequence has no cones under it.

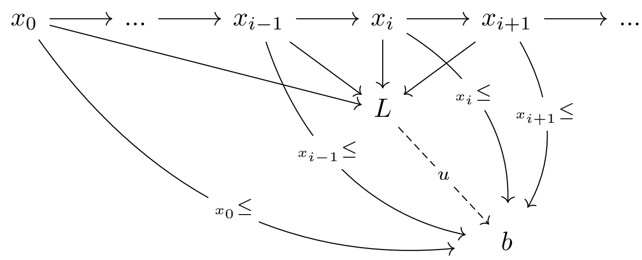
Definition. The **colimit** of a diagram in a category C $F : \mathsf{J} \to \mathsf{C}$ is a cone (L, ϕ) under F such that any other cone under F factors uniquely through L.



In other words, for any $\text{cone}(V, \lambda)$ under $F : \mathsf{J} \to \mathsf{C}$, there exists a unique morphism $u : L \to V$ such that $u\phi_x = \lambda_x : F(x) \to V$ for every $x \in \mathsf{J}$.

Returning to the previous example, if a diagram $F : \mathsf{Seq}_{\leq} \to (\mathbb{R}, \leq)$ has a colimit L, it means that there is a morphism $u : L \to b$ for any $\text{cone}(b,_{x_i} \leq)$.

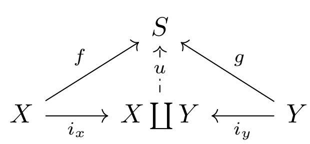
In the context of this category, this means that for any upper bound of b of the sequence $\{x_i\}_{i=0}^{\infty}$ that this diagram represents, $L \leq b$. Therefore, L is the least upper bound of $\{x_i\}_{i=0}^{\infty}$, and as this sequence is monotonic it converges to L.



Colimits are a powerful concept; many seemingly unrelated concepts may be interpreted as colimits with an appropriate choice of category and diagram. In an intuitive sense, colimits tend to "glue" objects together, though this is somewhat of an oversimplification. Here are some examples:

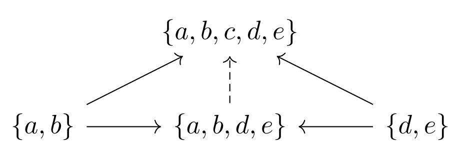
Coproducts

Given two objects X, Y in a category C, their coproduct $X \coprod Y$ is defined to be the colimit of a discrete diagram consisting of X and Y. The following figure illustrates this situation.



One can intuitively think of the coproduct of two objects as "adding" them together, and in fact, the coproduct is sometimes called the categorical sum for this reason. The following two examples are rather simple cases of coproducts.

1. The coproduct within Sub_S for two subsets of the parent set $U, V \subset S$ we have that their coproduct is the union of the two sets. This is because the only morphisms in this category are inclusions of subsets, so two subsets U, V with morphisms into a third subset W implies that W contains both U, V and therefore contains their union, so the inclusion of U and V into W always factors through $U \cup V$. If we set S to be the English alphabet, and $U = \{a, b\}$, $V = \{d, e\}$, and $W = \{a, b, c, d, e\}$ then the coproduct diagram looks like this.



2. Consider $\mathsf{Subg}_{\mathbb{Z}}$, the category of subgroups of $\mathbb{Z}$. Let our two objects be $n\mathbb{Z}$ and $m\mathbb{Z}$, as any subgroup of $\mathbb{Z}$ looks like $n\mathbb{Z}$ for some integer n. It follows that the coproduct of $n\mathbb{Z}$ and $m\mathbb{Z}$ will be $d\mathbb{Z}$ for some integer d. Again, by the definition of the coproduct, any morphisms from $m\mathbb{Z}$ and $n\mathbb{Z}$ into $k\mathbb{Z}$ must factor through $d\mathbb{Z}$. Said in another way, if $m\mathbb{Z}$ and $n\mathbb{Z}$ are subgroups of $k\mathbb{Z}$ then $d\mathbb{Z}$ must also be a subgroup of $d\mathbb{Z}$. It follows that $d = gcd(m, n)$.

Here are some more concrete examples.

1. Within Set the coproduct of sets X and Y is simply the disjoint union of those sets, hence the notation $\coprod$.

2. In Group this happens to be the **free product** described earlier. This can be seen as follows: Let G_1 and G_2 be groups with group homomorphisms $f_1 : G_1 \to K$, $f_1(g_1) = k_1$ and $f_2 : G_2 \to K$, $f_2(g_2) = k_2$. Additionally, suppose that there is a group H with homomorphisms $\iota_i : G_i \to H$ such that there is a unique homomorphism $u : H \to K$ for any K such that $u \circ \iota_i = f_i : H \to K$. Set K to be G_1 and f_1 to be the identity map. It follows that because $u\iota_1$ is the identity map, ι_1 must be injective, so G_1 is a subgroup of H. A similar argument may be made for $G_2 \subset H$. Therefore, we may set ι_i to be the inclusion map into H from G_i. Let $H = G_1 * G_2$. Then for any $f_i : G_i \to K$ we have the induced homomorphism $u : H \to K$ where $u \circ \iota_i(g_i) = f_i(g_i)$, so an element $a_1 a_2 ... n_1 n_2$ of H is sent to $f_1(a_1)f_2(a_2)...f_1(n_1)f_2(n_2)$. This is the only such homomorphism that commutes with the f_i homomorphisms because we have set ι_i to be the inclusion maps.

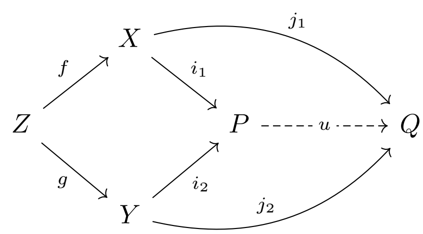
3. In Ab, the category of abelian groups, the coproduct of two groups is much simpler than it is in Group, as being confined to abelian groups forces the coproduct to be abelian itself. As a result, the coproduct of two abelian groups A and B is $A \bigoplus B$, the direct sum of A and B.

4. Interestingly, the coproduct within Top and Top_* are not the same. Within Top, the coproduct of two spaces is the disjoint union topology of those spaces, as one might expect given the first example[7]. However, within Top_* each space has a specified base point, and as the morphisms in this category are continuous maps that map base points to base points it follows that from setting the maps from X_* and Y_* into $(X \coprod Y)_*$ to be inclusion maps that the base point is shared between the two otherwise disjoint sets. In other words, $(X \coprod Y)_*$ is the disjoint union of X_* and Y_* with their base points identified, otherwise known as the wedge sum of X and Y.

[7]Again, the objects and morphisms of concrete categories are just sets and functions along with additional structure, so categorical structures within Set often provides insight into how those same structures will look in other concrete categories

Pushouts

A pushout is the colimit of a diagram of three objects, X, Y, Z with two morphisms $f : Z \to X$, $g : Z \to Y$ sharing a common domain out of one of the objects into the two others. Said in another way, it is the colimit of a diagram indexed as $\bullet \leftarrow \bullet \to \bullet$. Concisely, the pushout P along with morphisms i_1, i_2 makes the following diagram commute and any other object morphism triple (Q, j_1, j_2) that makes the diagram commute must factor through P via a unique morphism u.



Intuitively, one may think of the pushout P of two objects X, Y with morphisms $f : Z \to X$, $g : Z \to X$ as the object obtained by "glueing" X and Y together along the images $f(Z)$ and $g(Z)$. We will now explore some examples of pushouts.

1. Within Set, the pushout of two sets X, Y with functions $f : Z \to X$, $g : Z \to Y$ is the disjoint union of X and Y along with the finest equivalence relation such that $f(z) \sim g(z) \forall z \in Z$.

2. A particular kind of pushout known as an **adjunction space** is often used in topology. Suppose you have a topological space Y with a subspace Z and you want to create a new space P that looks like Y glued to another space X along Z. A natural way to do this is to find a continuous map $f : Z \to X$ and to then take the pushout of f and the inclusion of Z into Y. The resulting space is the disjoint union of X and Y with the quotient map identifying points in Z with their image in X. For example, let X and Y both be disks D^n, and f and g be the inclusions of S^{n-1} into ∂X and ∂Y respectively. Then P is the space achieved by glueing the two discs together along their boundary, S^n.

3. In the category of Groups pushouts are given by the free product with amalgamation. The free product with amalgamation is given by taking two groups, G and H, along with two monorphisms $\phi : F \to G$, $\psi : H \to H$ where F is an arbitrary set. The free product begins as normal, but with the added relation that $\phi(f)\psi^{-1}(f) = 1$. This enforces a relationship between the elements of G and H.

   The amalgamated free product can also be defined by letting N be the smallest subgroup of $G*H$ containing all elements of the form $\phi(f)\psi^{-1}(f)$. Then the amalgamated free product is given by the quotient group $(G * H)/N$

These last two examples are what is basically going on within the Van Kampen theorem, at least in categorical terms. Of course, the devil is in the details, so we finally close our brief jaunt through category theory to move on to proving the Van Kampen theorem.

Van Kampen's Theorem

Van Kampen's theorem is a statement which allows us to compute the fundamental group of a space which is comprised of several smaller spaces. Van Kampen's can be stated in a number of different ways, so two different statements of it will be given here. First we will discuss the version of the theorem which uses colimits, and then we will move to a slightly more intuitive statement of the theorem. They are equivalent, and both provide important perspectives on what this theorem means for our final construction.

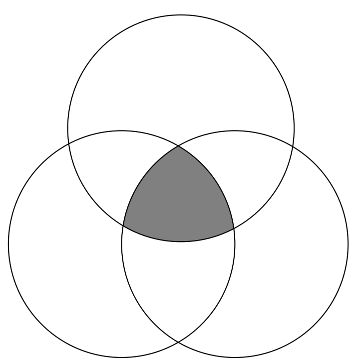
Van Kampen's Theorem in the language of colimits

Theorem. *Let X be path connected and choose a basepoint $x \in x$ Let $\mathcal{O}$ be a cover of X by path connected open subsets such that the intersection of finitely many subsets in $\mathcal{O}$ is again in $\mathcal{O}$ and x is in each $U \in \mathcal{O}$. Regard $\mathcal{O}$ as a category whose morphisms are the inclusions of subsets and observe that the functor $\pi_1(-, x)$, restricted to the spaces and maps in $\mathcal{O}$, gives a diagram $\pi_1|\mathcal{O} : \mathcal{O} \to G$ of groups. The group $\pi_1(X, x)$ is the colimit of this diagram. In symbols,*

$$\pi_1(X, x) \cong colim_{U\in\mathcal{O}}\pi_1(U, x)$$

Because of the importance of this theorem, it is important to go through it step by step. We begin with a space X which is path connected, and we chose a base point $x \in X$.

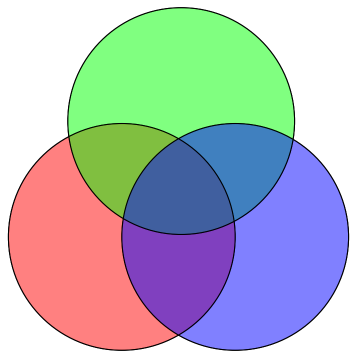
As an example, consider a space X which consists of 3 circles, where the base point x is in their intersection.



(1)

We will choose of find some open cover of X, written $\mathcal{O}$, which has the property that $\forall U_1, U_2, ..., U_n \in \mathcal{O}, U_1 \cap U_2 ... \cap U_n \in \mathcal{O}$, and the property that $x \in U \, \forall \, U \in \mathcal{O}$.

In our example, the picture of this might look like on open set in the cover for each circle in X and an open set for each of the intersections, 7 open sets in total.



(2)

We think of this cover as a category, where the objects are open sets in the cover and the morphisms between them are the inclusion maps. Then, we consider the fundamental group of each open set in the cover. This is the fundamental step in the theorem, we have found a collection of open sets which 'form' our larger space, and we consider the fundamental group of each piece. The goal of this theorem is to formalize the way in which the fundamental groups can be 'stitched' together to form the fundamental group of the whole space. To do this, we define a functor, a map between categories, which maps each open set in the cover to its fundamental group.

In the example we have been considering, the fundamental group of each piece is the trivial group, since each open set is simply connected.

This gives us a diagram, $\pi_1|\mathcal{O} : \mathcal{O} \to G$ of groups, and the fundamental group of X is the colimit of this diagram.

This is a somewhat non-intuitive result, and is the primary reason for the inclusion of the second statement of the theorem, which may be more clear, and which is certainly easier to use in basic computations.

Van Kampen's Theorem in the language of isomorphisms

Theorem. *If X is the union of path-connected open sets U_α each containing the basepoint $x \in X$ and if each intersection $U_\alpha \cap U_\beta$ is path-connected, then the homomorphism $\phi : *_\alpha \pi_1(U_\alpha) \to \pi_1(X)$is surjective. If in addition each intersection $U_\alpha \cap U_\beta \cap U_\gamma$ is path-connected, then the kernel of ϕ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1}$, and so ϕ induces an isomorphism $\pi_1(X) = *_\alpha \pi_1(U_\alpha)/N$.*

$i_{\alpha\beta}(\omega)$ refers to an element ω in the intersection $U_\alpha \cap U_\beta$ which, through the inclusion map, is being viewed as an element of X. Furthermore, the notation $*_\alpha \pi_1(U_\alpha)$ refers to the free product $U_{\alpha_1} * U_{\alpha_2} * ... * U_{\alpha_n}$.

This theorem makes it easier to see the 'big picture' of Van Kampen's theorem, that any element of the fundamental group of the larger space is constructed by going through loops in each of the pieces comprising it. To construct any loop in X with basepoint x, one can take a loop in any U_i or, with a bit of fiddling, concatenate and reduce loops going through several U_i.

Most theorems are best understood by going through a few examples of their application, and Van Kampen's is no exception.

Examples

1. **A Disk and a Loop**

   Let S be a circle and consider the space X formed by attaching a disk D along the boundary.

   The space X can be written as $(\partial D \cup S) \cup (D)$, a union of two open sets, and the intersection $(\partial D \cup S) \cap (D) = \partial D$ is path connected. We will compute the fundamental group $\pi_1(X, x)$ where x is some point in the intersection ∂D.

   Van Kampen's theorem tells us that:

   $$\pi_1(X) = *_\alpha \pi_1(U_\alpha)/N$$

   Since we only have two open sets in our cover,

   $$*_\alpha \pi_1(U_\alpha) = \pi_1(\partial D \cup S) * \pi_1(D) = \mathbb{Z} * \{1\} = \mathbb{Z}$$

   For the case when the cover consists of only 2 sets with a simply connected intersection, N reduces to $\pi_1(U_1 \cap U_2)$, so here it can be computed as

   $$N = \pi_1(\partial(D)) = \mathbb{Z}.$$

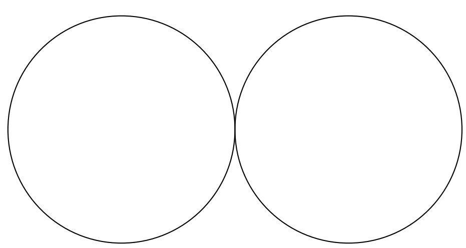
   $N = \pi_1(\partial(D)) = \mathbb{Z}$.

   Combining these two results tells us that:

   $$\pi_1(X) = *_\alpha \pi_1(U_\alpha)/N = \mathbb{Z}/\mathbb{Z} = 1, \text{ the trivial group.}$$

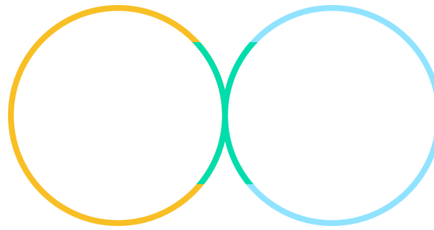
2. **A Figure 8**

   Consider a space X formed by two circles, connected at a single point x:

   (3)

   We will compute the fundamental group $\pi_1(X, x)$, by again decomposing this space into two open sets, U_1 and U_2. U_1 is comprised of the first circle, and a small piece extending into the second, and U_2 is the reverse.



Since U_1 and U_2 deformation retract to circles, they both have fundamental group $\mathbb{Z}$.

We can again use the simplification of N to compute

$$N = \pi_1(U_1 \cap U_2).$$

$U_1 \cap U_2$ deformation retracts to a single point, so it has the trivial fundamental group and we can compute:

$$\pi_1(X) = *_\alpha \pi_1(U_\alpha)/N = \mathbb{Z} * \mathbb{Z}/\{1\} = F_2$$

Here F_2 refers to the free group on two generators, the result of the free product of two copies of $\mathbb{Z}$.

3. **A Bouquet of Circles**

We can use an inductive process to find the fundamental group of a bouquet of n circles, all joined at a single point.

We begin with the result from the previous example, and consider adding another circle. We can define U_1 as the open set containing the original 2 circles, as well as a small open neighborhood of the new circle. Similarly, U_2 can be defined as the new circle, with a small open neighborhood of the old two. Then the computation follows almost exactly as before. $U_1 \cap U_2$ again deformation retracts to a single point, so it has the trivial fundamental group, and we know $\pi_1(U_1) = F_2 = \mathbb{Z} * \mathbb{Z}$, and $\pi_1(U_2) = \mathbb{Z}$

Hence,

$$\pi_1(X) = *_\alpha \pi_1(U_\alpha)/N = (F_2 * \mathbb{Z})/\{1\} = (\mathbb{Z} * \mathbb{Z} * \mathbb{Z})/\{1\} = F_3$$

Following this same procedure allows us to calculate that the fundamental group of a bouquet of n circles will always be F_n, the free group on n generators.

Final Construction

We now endeavor to construct a space X with fundamental group of G, where G is finitely presented. Let G be presented by

$$\langle g_1, \dots, g_n | r_1, \dots, r_m \rangle.$$

We begin with the space R_n, the rose with n petals, whose fundamental group is F_n, the free group on n letters. F_n has no relations, so we need to add them in. Considering r_1 as a reduced word with k letters, we define a map $f_1 : S^1 \to R_n$ by splitting S^1 into k equal pieces, and send the first piece around the first letter of r_1, and the ith section to the ith letter of r_1 (with respect to the labels of R_n). We attach the disk D^2 along this map f_1, and call the space after attaching the disk X_1. Since $\pi_1(D^2) = 1$, Van Kampen's Theorem implies that

$$\pi_1(X_1) \cong \langle g_1, \dots, g_n | r_1 \rangle.$$

Now we do an inductive step. Assume X_i has fundamental group $\langle g_1, \dots, g_n | r_1, \dots, r_{i-1} \rangle$. We define $f_i : S^1 \to R_n$ in the same way we did for f_1, using the word r_i. There is a natural map $q_i : R_n \to X_i$, which is similar to the inclusion, although R_n may have relations which prevent injectivity. We attach a disk by the map $q_i \circ f_i$ to define X_{i+1}, which has fundamental group

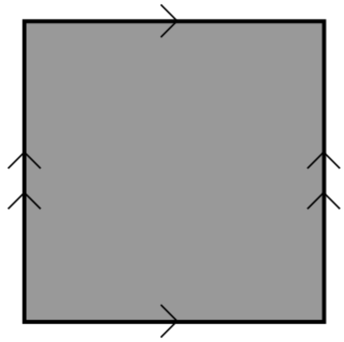
$$\pi_1(X_{i+1}) \cong \langle g_1, \dots, g_n | r_1, \dots, r_i \rangle.$$

Letting $X = X_m$ yields our result.

Now, we will give some examples of some spaces with fundamental groups of common finitely presented groups.

- Let C_n be the cyclic group of order n. To obtain the space X with $\pi_1(X) \cong C_n$, we attach a disk to S^1 along the map $f(e^{2\pi i x}) = e^{2\pi i x \cdot n}$. The resulting space looks like a disk, but on the boundary each point is identified with $n-1$ other points which are spaced evenly around the circle.

- The Klein 4 Group is presented by $\langle a, b | a^2, b^2, (ab)^2 \rangle$. So we attach 3 disks to R_2. We first attach the maps killing a^2 and b^2. If X_{C_2} is the space obtained from the previous example using C_2, after this attachment we have a space homeomorphic to $X_{C_2} \wedge X_{C_2}$ with fundamental group $C_2 * C_2$. We then attach the map going around $abab$, which kills the remaining free part of our group leaving us with the Klein 4 Group.



[9]

- Finally, we'll give an example of a space X with $\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}$ using our construction. This is presented by $\langle a, b | a^{-1}b^{-1}ab \rangle$. We attach a disk via the obvious map which gives us our space. This space X turns out to be homeomorphic to the torus! If we think of the attaching map as a map of the square[8] with each edge mapping to a different letter, we can reverse the arrows of the a^{-1} and b^{-1} edges to relabel them a and b. Then we have the standard identification of sides of the square inducing the torus.

[8]The square is homeomorphic to S^1.

References

[1] Mladen Bestvina. *Folding graphs and applications, d'après Stallings*. Available at http://www.math.utah.edu/~bestvina/folding.pdf.

[2] Tai-Danae Bradley. *Limits and Colimits, Part 1 (Introduction)*. Available at https://www.math3ma.com/blog/limits-and-colimits-part-1 (2021).

[3] Joshua Gensler. *An Application of the Van Kampen Theorem*. Available at http://www.math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Gensler.pdf (2021).

[4] Allen Hatcher. *Algebraic topology*. Cambridge: Cambridge Univ. Press, 2000. URL: https://cds.cern.ch/record/478079.

[5] J. P. May. *A Concise Course in Algebraic Topology*. 1st ed. University of Chicago Press, 1999.

[6] J.R. Munkres. *Topology*. Featured Titles for Topology. Prentice Hall, Incorporated, 2000. ISBN: 9780131816299. URL: https://books.google.com/books?id=XjoZAQAAIAAJ.

[7] Andrew Putman. *A quick proof of the Seifert–Van Kampen theorem*. Available at https://www3.nd.edu/~andyp/notes/SeifertVanKampen.pdf (2021).

[8] Emily Riehl. *Category Theory in Context*. 1st ed. Dover, 2016.

[9] Tim Speer. *What is a Torus?* Available at https://steemit.com/math/@timspeer/what-is-a-torus.

[10] Robin Truax. *An Introduction to Category Theory*. Available at http://web.stanford.edu/~truax/notes/Category_Theory.pdf (2021).