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A (very) Rough Tour of Sheaves and Cech Cohomology

MATH 5320

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1 Introduction

We provide exposition on Sheaf Theory based on Vakil's text [Vak15], and then proceed to introduce Riemann surfaces and Čech cohomology from Miranda's text [Mir95].

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2 Sheaves

Sheaves

Sheaves are objects defined on topological spaces. The goal of a sheaf is to encode local information on the space. To define a sheaf though, we must first define the presheaf.

Definition *Presheaves*

Let X be a topological space. A presheaf $\mathscr{F}$ on X is a mapping encoded by the following:

1) For each open set $U \subseteq X$, there is a set $\mathscr{F}(U)$

2) For each inclusion of open sets $U \subseteq V$, there is a function

$$res_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$$

The two following conditions must also hold:

· For all open $U \subseteq X$, $res_{U,U} = id_{\mathscr{F}(U)}$.

· If $U \subseteq V \subseteq W$ are open sets, then

$$res_{W,U} = res_{V,U} \circ res_{W,V}$$

The presheaf $\mathscr{F}$ is essentially the collection of all the information above. When defining a presheaf on a space X, one must specify both the sets $\mathscr{F}(U)$ for every open set $U \subseteq X$ and the maps $res_{V,U}$ for each inclusion $U \subseteq V$. The elements of $\mathscr{F}(U)$ are called sections over U. The maps $res_{V,U}$ will frequently be restiction maps, hence the notation.

For those familiar with category theory, a presheaf $\mathscr{F}$ on X is simply a contravariant functor $\mathscr{F} : \mathscr{C} \to Sets$ where $\mathscr{C}$ is the category of open sets in X.

Notes on Presheaves

The definition of a presheaf that we gave is actually the definition for a presheaf of *sets*. The above definition can be extended to different classes of objects. For example, a presheaf of *groups* would have the exact same definition given above except for each open $U \subseteq X$, $\mathscr{F}(U)$ is now a group and for each inclusion $U \subseteq V$, $res_{V,U}$ is now a group homomorphism. We can similarly define presheaves of rings, Abelian groups, R-modules, etc.

Examples of presheaves

Continuous Functions

Let X, Y be topological spaces. For each open $U \subseteq X$, define:

$$\mathscr{F}(U) = \{f : U \to Y \mid f \textit{ is continuous}\}$$

and for each inclusion $U \subseteq V$, define $res_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$ by:

$$res_{V,U}(f) = f|_U$$

Claim: $\mathscr{F}$ is a presheaf.

Proof. Let $U \subseteq X$ be open. Then $res_{U,U}(f) = f|_U = f$ since $f : U \to Y$. Thus, $res_{U,U} = id_{\mathscr{F}(U)}$

Now let $U \subseteq V \subseteq W$. Then we have:

$$res_{W,U}(f) = f|_U = (f|_V)|_U = res_{V,U} \circ res_{W,V}(f)$$

So $res_{W,U} = res_{V,U} \circ res_{W,V}$.

$\therefore \mathscr{F}$ is a presheaf on X.

■

In the above example, $\mathscr{F}$ is a presheaf of sets. If Y has a ring structure (e.g. $Y = \mathbb{R}$), then $\mathscr{F}$ would be a presheaf of rings since $\mathscr{F}(U)$ defines a ring of functions and the restriction maps $res_{V,U}$ would define ring homomorphisms.

Here's another example:

The Constant Presheaf

Let S be any set and X a topological space. We define the constant presheaf $\underline{S}^{pre}$ on X by setting $\underline{S}^{pre}(U) = S$ for all open $U \subseteq X$ and $res_{V,U} = id_S$ for all inclusions $U \subseteq V$.

Now we can define sheaves on a topological space X.

Definition *Sheaves*

Let $\mathscr{F}$ be a presheaf on a topological space X. $\mathscr{F}$ is a sheaf on X if the following two conditions hold:

1) **Identity Axiom** Let $U \subseteq X$ be an open set and $\bigcup_{\alpha \in J} U_\alpha = U$ an open cover of U. If $f_1, f_2 \in \mathscr{F}(U)$ are two sections such that $res_{U,U_\alpha}(f_1) = res_{U,U_\alpha}(f_2)$ for all $\alpha \in J$, then $f_1 = f_2$.

2) **Gluability Axiom** Let $U \subseteq X$ be an open set and $\bigcup_{\alpha \in J} U_\alpha = U$ an open cover of U. For each $\alpha \in J$, let $f_\alpha \in \mathscr{F}(U_\alpha)$. If, for all $\alpha, \beta \in J$, we have:

$$res_{U_\alpha, U_\alpha \cap U_\beta}(f_\alpha) = res_{U_\beta, U_\alpha \cap U_\beta}(f_\beta)$$

Then there exists a section $f \in \mathscr{F}(U)$ such that $res_{U,U_\alpha}(f) = f_\alpha$ for all $\alpha \in J$.

Example and non-example of a sheaf

Let's consider the two examples of presheaves that we gave before.

Continuous Functions

Continuous functions will form a sheaf on X.

Proof. Let $\mathscr{F}$ be the presheaf on X given by continous functions $f : X \to Y$, as before. To show that $\mathscr{F}$ is a sheaf, we just need to check the Identity Axiom and the Gluability Axiom.

1) **Identity** Let U be open and $\bigcup_{\alpha \in J} U_\alpha = U$ an open cover of U. Let $f_1, f_2 \in \mathscr{F}(U)$ be two sections such that $res_{U,U_\alpha}(f_1) = res_{U,U_\alpha}(f_2)$ for all $\alpha \in J$. Then $f_1|_{U_\alpha} = f_2|_{U_\alpha}$ for all $\alpha \in J$. Let $x \in U$. Then $x \in U_\alpha$ for some $\alpha \in J$. So we have that

$$f_1(x) = f_1|_{U_\alpha}(x) = f_2|_{U_\alpha}(x) = f_2(x)$$

Thus, $f_1 = f_2$.

2) **Gluability** Again, let U be open and $\bigcup_{\alpha \in J} U_\alpha = U$ an open cover of U. Let $f_\alpha \in \mathscr{F}(U_\alpha)$ for each $\alpha \in J$ so that for any pair $\alpha, \beta \in J$, we have

$$res_{U_\alpha, U_\alpha \cap U_\beta}(f_\alpha) = res_{U_\beta, U_\alpha \cap U_\beta}(f_\beta)$$

Then we have

$$f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$$

for any $\alpha, \beta \in J$.

Define $f : U \to Y$ by $f(x) = f_\alpha(x)$ when $x \in U_\alpha$.

f is well defined because of the condition that $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$ for any $\alpha, \beta \in J$. Furthermore, we have that $f|_{U_\alpha} = f_\alpha$ is continuous for every restriction corresponding to $\alpha \in J$. Thus, f is also continuous and $f \in \mathscr{F}(U)$. f satisfies the condition for the Gluability Axiom and so we are done.

$\therefore \mathscr{F}$ is a sheaf on X. ■

Constant presheaf (Non-example)

It turns out that $\underline{S}^{pre}$ is not a sheaf in general. We will not go into details but if X is a two-point discrete space and S has at least two elements, then $\underline{S}^{pre}$ will fail the gluability axiom.

To turn the constant presheaf into a sheaf, we need to slightly adjust the definition using the fact that continuous functions form a sheaf.

Definition Constant Sheaf

Let S be a set, X a topological space and define $\underline{S}$ to be the sheaf formed by the continuous functions from $X \to S$ where S is equipped with the discrete topology. $\underline{S}$ is called the constant sheaf.

Another way to think about $\underline{S}$ is the following: the constant sheaf $\underline{S}$ is constructed by all the maps $X \to S$ which are locally constant.

Here is another important sheaf.

Proposition Skyscraper Sheaf

Let X be a topological space, $p \in X$, and S a set. Let $i_p : \{p\} \to X$ denote the inclusion map. For an open $U \in X$, define the map $i_{p,*}S(U)$ as follows: if $p \in U$, then $i_{p,*}S(U) = S$. Otherwise, $i_{p,*}S(U) = \{e\}$ where $\{e\}$ is some fixed singleton. $i_{p,*}S$ defines a sheaf on X and called the skyscraper sheaf.

Verifying that the skyscraper sheaf does indeed define a sheaf is slightly painful so we will skip the proof. However, the skyscraper sheaf can be defined in a different, arguably more intuitive way than as above. To give the alternative definition, we first define the sheaf of sections on a map.

Proposition Sheaf of sections on a map

Let $\mu : Y \to X$ be a continuous map. To each open $U \subseteq X$, define $\mathscr{F}(U)$ by

$$\mathscr{F}(U) = \{s : U \to Y | s \text{ is continuous and } \mu \circ s = id_U\}$$

$\mathscr{F}$ defines a sheaf on X.

Proof. The restriction maps are $res_{V,U}(s) = s|_U$. Let $U \subseteq X$ be an open set and $\bigcup_{\alpha \in S} U_\alpha = U$ an open cover of U.

1) (Identity) Let $f, g \in \mathscr{F}(U)$ such that $res_{U,U_\alpha} f = res_{U,U_\alpha} g$ for all $\alpha \in S$. Let $x \in U$. Then $x \in U_\alpha$ for some $\alpha \in S$. So we have

$$res_{U,U_\alpha} f(x) = res_{U,U_\alpha} g(x)$$
$$\to f|_{U_\alpha}(x) = g|_{U_\alpha}(x)$$
$$\to f(x) = g(x)$$

Thus, $f = g$.

2) (Gluability) To each $\alpha \in S$, let $f_\alpha \in \mathscr{F}(U_\alpha)$ such that $res_{U_\alpha, U_\alpha \cap U_\beta} f_\alpha = res_{U_\beta, U_\alpha \cap U_\beta} f_\beta$ (i.e. $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$).

Define $f(x) = f_\alpha(x)$ when $x \in U_\alpha$. The condition implies that f is well defined and continuous on U. By definition of f, we have $res_{U,U_\alpha} f = f_\alpha$. Furthermore, we have

$$\mu \circ f(x) = \mu \circ f_\alpha(x) = id_{U_\alpha}(x) = id_U(x)$$

So $f \in \mathscr{F}(U)$.

$\therefore \mathscr{F}$ is a sheaf on X. ■

We now define one more important class of sheaves.

Definition Pushforward Sheaf

Let $\pi : X \to Y$ be a continuous map and $\mathscr{F}$ a presheaf on X. The pushforward of $\mathscr{F}$ by π is the collection of maps $\pi_*\mathscr{F}(V) = \mathscr{F}(\pi^{-1}(V))$ where $V \subseteq Y$ is open.

Proposition

The pushforward $\pi_*\mathscr{F}$ defines a (pre)sheaf on Y if $\mathscr{F}$ is a (pre)sheaf.

Proof. First we verify that the pushforward defines a presheaf if $\mathscr{F}$ is a presheaf.

$\underline{Presheaf}$

Let $U \subseteq V \subseteq W \subseteq Y$ be open sets.

(1) We have

$$res_{U,U} = res_{\pi_{\pi^{-1}(U),\pi^{-1}(U)}} = id_{\mathscr{F}(\pi^{-1}(U)} = id_{\pi_*\mathscr{F}(U)}$$

Note: The first equality is between the restriction map in $\pi_*\mathscr{F}$ and the restriction map in $\mathscr{F}$.

(2) We have that $\pi^{-1}(U) \subseteq \pi^{-1}(V) \subseteq \pi^{-1}(W)$. So then

$$res_{W,U} = res_{\pi^{-1}(W),\pi^{-1}(U)}$$

$$= res_{\pi^{-1}(V),\pi^{-1}(U)} \circ res_{\pi^{-1}(W),\pi^{-1}(V)}$$

$$= res_{V,U} \circ res_{W,V}$$

Thus, $\pi_*\mathscr{F}$ is a presheaf.

$\underline{Sheaf}$

Let $U \subseteq X$ be open and $\bigcup_{\alpha \in S} U_\alpha = U$ an open cover of U.

(1) (Identity) Let $f, g \in \pi_*\mathscr{F}(U)$ such that $res_{U,U_\alpha} f = res_{U,U_\alpha} g$ for all $\alpha \in S$. Then we have that

$$res_{\pi^{-1}(U),\pi^{-1}(U_\alpha)} f = res_{\pi^{-1}(U),\pi^{-1}(U_\alpha)} g$$

for all $\alpha \in S$. Observe that $\bigcup_{\alpha \in S} \pi^{-1}(U) = \pi^{-1}(U)$ is an open cover of $\pi^{-1}(U)$. Since $\mathscr{F}$ is a sheaf, it satisfies the identity axiom and thus, $f = g$.

(2) (Gluability) To each $\alpha \in S$, let $f_\alpha \in \pi^{-1}\mathscr{F}(U_\alpha)$ such that

$$res_{U_\alpha, U_\alpha \cap U_\beta} f_\alpha = res_{U_\beta, U_\alpha \cap U_\beta} f_\beta$$

for all $\alpha, \beta \in S$. $\bigcup_{\alpha \in S} \pi^{-1}(U) = \pi^{-1}(U)$ is an open cover of $\pi^{-1}(U)$ and the f_α satisfy the gluability axioms for $\mathscr{F}(\pi^{-1}(U))$, so there exists an $f \in \mathscr{F}(\pi^{-1}(U)) = \pi_* \mathscr{F}(U)$ such that $res_{\pi^{-1}(U), \pi^{-1}(U_\alpha)} f = f_\alpha$ for all $\alpha \in S$. This implies that $res_{U, U_\alpha} f = f_\alpha$ for all $\alpha \in S$ and so the gluability axiom is satisfied.

$\therefore \pi_* \mathscr{F}$ is a sheaf on Y. ■

Using the pushforward sheaf, we can give an alternative definition for the skyscraper sheaf.

Definition Skyscraper Sheaf

Let X be a topological space, $p \in X$, S a set, $\underline{S}$ the constant sheaf on $\{p\}$, and $i_p : \{p\} \to X$ the inclusion map. The skyscraper sheaf is the pushforward of $\underline{S}$ by i_p.

With the above definition, it's now clear why we used the notation $i_{p,*}S$ to denote the skyscraper sheaf.

Stalks/Germs

We now define a central subject in sheaf theory: Stalks and Germs.

Definition Stalks and Germs

Let X be a topological space, $p \in X$, and $\mathscr{F}$ a presheaf on X. A germ at p is a pair (f, U) where U is an open neighborhood of p and f is a section over U (i.e. $f \in \mathscr{F}(U)$). Two germs (f, U) and (g, V) at p are declared equivalent, denoted $(f, U) \sim (g, V)$, if there exists an open neighborhood $W \subseteq U \cap V$ of p such that $res_{U,W} f = res_{V,W} g$.

The stalk at p, denoted $\mathscr{F}_p$, is the collection of all germs at p. In other words, $\mathscr{F}_p$ is given by:

$$\mathscr{F}_p = \{(f, U) \mid p \in U\} / \sim$$

The definition for stalks and germs of a sheaf is identical.

The type of sheaf that $\mathscr{F}$ defines will determine the structure of the stalks. For example, if $\mathscr{F}$ is a sheaf of groups then the stalks will be groups.

Sheaf Morphisms

With any algebraic object, we want to understand the maps between those objects. Sheaves are no exception. We now define morphisms of sheaves.

Definition *Morphism of Sheaves*

Let X be a topological space and $\mathscr{F}, \mathscr{G}$ sheaves of sets on X. A morphism of sheaves from $\mathscr{F}$ to $\mathscr{G}$, denoted $\phi : \mathscr{F} \to \mathscr{G}$, is a collection of maps $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ defined for each open $U \subseteq X$ such that for any inclusion $U \subseteq V$, we have

$$res_{V,U} \circ \phi(V) = \phi(U) \circ res_{V,U}$$

In other words, the following diagram commutes for all inclusions $U \subseteq V$:

$$\begin{array}{ccc} \mathscr{F}(V) & \xrightarrow{\phi(V)} & \mathscr{G}(V) \\ {\scriptstyle res_{V,U}} \downarrow & & \downarrow {\scriptstyle res_{V,U}} \\ \mathscr{F}(U) & \xrightarrow[\phi(U)]{} & \mathscr{G}(U) \end{array}$$

Remark Just as we did for the definition of sheaves, what we've actually defined above is a morphism of sheaves when the sheaves $\mathscr{F}$ and $\mathscr{G}$ are sheaves of sets. We can define morphisms of sheaves of any type however. $\mathscr{F}$ and $\mathscr{G}$ must be the same kind of sheaf and the maps $\phi(U)$ must be appropriate morphisms matching the type of object that the sheaves define. So if $\mathscr{F}$ and $\mathscr{G}$ are sheaves of groups, then the $\phi(U)$ must be group homomorphisms. Similarly, if $\mathscr{F}$ and $\mathscr{G}$ are sheaves of rings, then the $\phi(U)$ must be ring homomorphisms.

To finish our introduction to sheaves, we give one last definition.

Definition Sheaf Hom

Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on a space X. For an open set $U \subseteq X$, define

$$Hom(\mathscr{F}, \mathscr{G})(U) = \{\phi : \mathscr{F}|_U \to \mathscr{G}|_U \mid \phi \textit{ is a sheaf morphism}\}$$

The sheaf Hom of $\mathscr{F}$ and $\mathscr{G}$, denoted $Hom(\mathscr{F}, \mathscr{G})$, is the collection of all $Hom(\mathscr{F}, \mathscr{G})(U)$.

Proposition
If $\mathscr{F}$ and $\mathscr{G}$ are sheaves on a space X, then the sheaf Hom of $\mathscr{F}$ and $\mathscr{G}$ defines a sheaf on X.

Proof. For a given morphism $\phi : \mathscr{F} \to \mathscr{G}$ and open $U \subseteq X$, define the restriction of ϕ to U, denoted $\phi|_U$, as $\phi|_U(W) = \phi(W)$ for open $W \subseteq U$. Note that $\phi|_U$ defines a morphism of sheaves from $\mathscr{F}|_U$ to $\mathscr{G}|U$. Let $H(U) = Hom(\mathscr{F}, \mathscr{G})(U)$. Defining the restriction maps from $H(V) \to H(U)$ by $\phi \mapsto \phi|_U$, we see that the sheaf Hom is a presheaf on X. Now we check the sheaf axioms.

Let $U \subseteq X$ be open and $\bigcup_{\alpha \in S} U_\alpha = U$ an open cover of U.

(1) (Identity) Let $\phi, \psi \in H(U)$ such that $\phi|_{U_\alpha} = \psi|_{U_\alpha}$ for all $\alpha \in S$. We want to show that $\phi(W) = \psi(W)$ for all open $W \subseteq U$. Set $W_\alpha = W \cap U_\alpha$. Observe that $\bigcup_{\alpha \in S} W_\alpha = W$ is an open cover of W. Let $f \in \mathscr{F}(W)$. We have that $W_\alpha \subseteq U_\alpha$, so

$$\phi|_{U_\alpha}(W_\alpha) = \psi|_{U_\alpha}(W_\alpha)$$

for all $\alpha \in S$.

Set $g_1 = \phi(W)(f)$ and $g_2 = \psi(W)(f)$. Then for each α, we have

$$res_{W,W_\alpha} g_1 = [res_{W,W_\alpha} \circ \phi(W)](f)$$
$$= [\phi(W_\alpha) \circ res_{W,W_\alpha}](f)$$
$$= [\psi(W_\alpha) \circ res_{W,W_\alpha}](f)$$
$$= [res_{W,W_\alpha} \circ \psi(W)](f)$$
$$= res_{W,W_\alpha} g_2$$

So by the identity axiom of $\mathscr{G}$, $g_1 = g_2$. Thus, $\phi(W) = \psi(W)$ for all open $W \subseteq U$ and so $\phi = \psi$.

(2) (Gluability) To each $\alpha \in S$, let $\phi_\alpha \in H(U_\alpha)$ such that

$$\phi_\alpha|_{U_\alpha \cap U_\beta} = \phi_\beta|_{U_\alpha \cap U_\beta}$$

for all $\alpha, \beta \in S$. To each open $W \subseteq U$, we wish to define $\phi(W) : \mathscr{F}(W) \to \mathscr{G}(W)$ such that ϕ defines a morphism of sheaves and $\phi|_{U_\alpha} = \phi_\alpha$ for each $\alpha \in S$. Fix $W \subseteq U$ and a section $f \in \mathscr{F}(W)$. Set $f_\alpha = res_{W,W_\alpha} f$ and $g_\alpha = \phi_\alpha(W_\alpha)(f_\alpha)$ for each α. Note that $W_\alpha \cap W_\beta \subseteq U_\alpha \cap U_\beta$ for any $\alpha, \beta \in S$, so we have

$$\phi_\alpha(W_\alpha \cap W_\beta) = \phi_\beta(W_\alpha \cap W_\beta)$$

Thus,

$$res_{W_\alpha, W_\alpha \cap W_\beta}(g_\alpha) = res_{W_\alpha, W_\alpha \cap W_\beta} \circ \phi_\alpha(W_\alpha)(f_\alpha)$$
$$= \phi_\alpha(W_\alpha \cap W_\beta) \circ res_{W_\alpha, W_\alpha \cap W_\beta}(f_\alpha)$$
$$= \phi_\beta(W_\alpha \cap W_\beta) \circ res_{W_\alpha, W_\alpha \cap W_\beta}(f_\alpha)$$
$$= \phi_\beta(W_\alpha \cap W_\beta) \circ res_{W_\alpha, W_\alpha \cap W_\beta} \circ res_{W,W_\alpha}(f)$$
$$= \phi_\beta(W_\alpha \cap W_\beta) \circ res_{W, W_\alpha \cap W_\beta}(f)$$
$$= \phi_\beta(W_\alpha \cap W_\beta) \circ res_{W_\beta, W_\alpha \cap W_\beta} \circ res_{W,W_\beta}(f)$$
$$= \phi_\beta(W_\alpha \cap W_\beta) \circ res_{W_\beta, W_\alpha \cap W_\beta}(f_\beta)$$
$$= res_{W_\beta, W_\alpha \cap W_\beta} \circ \phi_\beta(W_\beta)(f_\beta)$$
$$= res_{W_\beta, W_\alpha \cap W_\beta}(g_\beta)$$

So then by the gluability axiom in $\mathscr{G}$, there exists a section $g \in \mathscr{G}(W)$ such that $res_{W,W_\alpha}(g) = g_\alpha$ for all $\alpha \in S$. Define $\phi(W)(f) = g$. We now have that $res_{W,W_\alpha} \circ \phi(W) = \phi_\alpha(W_\alpha) \circ res_{W,W_\alpha}$ for all $\alpha \in S$. To show that $\phi|_{U_\alpha} = \phi_\alpha$, let $W \subseteq U_\alpha$ be open. Then $W_\alpha = W$. So

$$res_{W,W_\alpha} \circ \phi(W) = \phi_\alpha(W_\alpha) \circ res_{W,W_\alpha}$$
$$\to res_{W,W} \circ \phi(W) = \phi_\alpha(W) \circ res_{W,W}$$
$$\to id_{\mathscr{G}(W)} \circ \phi(W) = \phi_\alpha(W) \circ id_{\mathscr{F}(W)}$$
$$\to \phi(W) = \phi_\alpha(W)$$

Thus, $\phi|_{U_\alpha} = \phi_\alpha$. We now know that the following diagram commutes:

$$\begin{CD}
\mathscr{F}(W) @>{\phi(W)}>> \mathscr{G}(W) \\
@V{res_{W,W_\alpha}}VV @VV{res_{W,W_\alpha}}V \\
\mathscr{F}(W_\alpha) @>>{\phi(W_\alpha)}> \mathscr{G}(W_\alpha)
\end{CD}$$

for all open $W \subseteq U$ and $\alpha \in S$. We just need to check that ϕ actually defines a morphism of sheaves now. Let $W \subseteq V \subseteq U$ be open sets. We want to show that the following diagram commutes:

$$\begin{CD}
\mathscr{F}(V) @>{\phi(V)}>> \mathscr{G}(V) \\
@V{res_{V,W}}VV @VV{res_{V,W}}V \\
\mathscr{F}(W) @>>{\phi(W)}> \mathscr{G}(W)
\end{CD}$$

First observe that $W_\alpha \subseteq V_\alpha \subseteq V$. So then the following diagram commutes:

$$\begin{CD}
\mathscr{F}(V) @>{\phi(V)}>> \mathscr{G}(V) \\
@V{res_{V,W}}VV @VV{res_{V,W}}V \\
\mathscr{F}(V_\alpha) @>>{\phi(V_\alpha)}> \mathscr{G}(V_\alpha) \\
@V{res_{V_\alpha,W_\alpha}}VV @VV{res_{V_\alpha,W_\alpha}}V \\
\mathscr{F}(W_\alpha) @>>{\phi(W_\alpha)}> \mathscr{G}(W_\alpha)
\end{CD}$$

So let $f \in \mathscr{F}(V)$. Let $g_1 = res_{V,W} \circ \phi(V)(f)$ and $g_2 = \phi(W) \circ res_{V,W}(f)$. Then we have

$$res_{W,W_\alpha}(g_2) = (res_{W,W_\alpha} \circ \phi(W)) \circ res_{V,W}(f)$$
$$= \phi(W_\alpha) \circ res_{W,W_\alpha} \circ res_{V,W}(f)$$
$$= \phi(W_\alpha) \circ res_{V,W_\alpha}(f)$$
$$= res_{V,W_\alpha} \circ \phi(V)(f)$$
$$= res_{W,W_\alpha} \circ res_{V,W} \circ \phi(V)(f)$$
$$= res_{W,W_\alpha}(g_1)$$

So by the identity axiom in $\mathcal{G}$, we have $g_1 = g_2$. Thus, $res_{V,W} \circ \phi(V) = \phi(W) \circ res_{V,W}$ and ϕ defines a morphism of sheaves. So the sheaf Hom satisfies the gluability axiom.

$\therefore$ the sheaf Hom is a sheaf. ■

This concludes the section on sheaves.

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3 Riemann Surfaces - Basic Definitions

We use much of the following theory on Riemann surfaces and Čech cohomology from Rick Miranda's book [Mir95].

1-manifolds are defined as sets that can be covered by images of charts mapping to real space. If one changed the domains of these diffeomorphisms to the complex plane, we get **Riemann surfaces**.

We take the mappings associated with zooming into a manifold to get complex manifolds to be **charts** $\phi : U \to \mathbb{C}$ where U is an open subset of topological space X. In order to define Riemann surfaces embedded outside of complex space, we only require ϕ to be bicontinuous onto its image as opposed to bi-smooth.

Between charts $\phi : U \to \phi(U)$ and $\psi : V \to \psi(V)$, we ensure compatibility by requiring that, if U and V are not disjoint, $U \cap V$ is open in X and $\phi \circ \psi^{-1} : \psi(U \cap V) \to \mathbb{C}$ is holomorphic.

3.1 Holomorphic and Meromorphic functions

Definition 3.1. A function f on Riemann surface X to $\mathbb{C}$ is **holomorphic** at $x_0 \in X$if there exists a chart defined near x_0 $\phi : U \to \mathbb{C}$, where $f \circ \phi^{-1}$ is holomorphic near $\phi(x_0)$.

We note a property of holomorphic functions and an interesting example.

Theorem 3.2. *If X is a Riemann surface and f, g are holomorphic functions on an open subset W of X, then $f + g$ and $f \cdot g$ are holomorphic on X. This makes the set of holomorphic functions on W a $\mathbb{C}$-algebra on X. That is, this set of functions is a vector space equipped with product $f, g \mapsto f \cdot g$. We denote this algebra $\mathcal{O}_X(W)$ or $\mathcal{O}(W)$ (suggestively for the theory of sheaves). If $g \neq 0$ at a point, then f/g is holomorphic at that point.*

A similar analogy of definitions can be made for maps with poles, essential singularities, or removable singularities.
The theorem proving that the set of holomorphic functions on an open subset W of Riemann surface X is a $\mathbb{C}$-algebra is also true for meromorphic functions. We can similarly define the order of a zero/pole using the same method.

Definition 3.3. Let $f : X \to \mathbb{C}$ be a complex-valued function on Riemann surface X. Then if we have a chart $\phi : U \to \mathbb{C}$ for X at point p, $f \circ \phi^{-1}$ has a **Laurent series expansion**

$$f \circ \phi^{-1}(z) = \sum_{n=-\infty}^{\infty} c_n (z - \phi(p))^n$$

When we deal with nonzero meromorphic functions, the **order** of f at p, $ord_p(f)$, will be finite and well defined.

4 Čech Cohomology

We first apply some notation changes and an equivalent definition of sheaves. From here, pre-sheaves and sheaves will be denoted by the letters $\mathcal{F}, \mathcal{G}$, and the restriction maps will be denoted $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ given $V \subset U$. We will also often omit restriction maps to make the notation as legible as possible. Given a pre-sheaf $\mathcal{F}$, we say it satisfies the **sheaf axiom** over a set X if for a cover $\mathcal{U}$ of an open set U, a section $f_\alpha \in \mathcal{F}(U_\alpha)$ for every $U_\alpha \in \mathcal{U}$, and for α, β in the index set J, $\rho_{U_\alpha \cap U_\beta}^{U_\alpha}(f_\alpha) = \rho_{U_\alpha \cap U_\beta}^{U_\beta}(f_\beta)$, then there exists a unique $f \in \mathcal{F}(U)$ restricting to f_α on each U_α. Note that this is equivalent to having both the gluability axiom and the identity axiom hold.

Definition 4.1. Let X be a Riemann surface. Let $D : X \to \mathbb{Z}$ be a map. D is called a **divisor** if for all points except on a discrete set, which we call the support, $D(p) = 0$.

Note that if X is compact, we immediately have that the support of D is finite.

Example 4.2. Define the following sheaf on X: given an open set U, let

$$\mathcal{O}_X[D](U) = \{\text{meromorphic functions on } U, f : U \to \mathbb{C}_\infty \mid \text{ord}_p f \geq D(p) \text{ for all } p \in U\}$$

The above is an abelian group under addition, so $\mathcal{O}_X[D]$ is a presheaf of groups, and in fact a sheaf. We verify the sheaf axiom for this presheaf. Suppose we have an open subset of X, U and a covering $\{U_i\}$ of U. Suppose on each U_i we have $f_i \in \mathcal{O}_X[D](U_i)$ satisfying agreement on restrictions

$$\rho_{U_i \cap U_j}^{U_i}(f_i) = \rho_{U_i \cap U_j}^{U_j}(f_j)$$

for all i, j. Define a function $f : U \to \mathbb{C}$ as follows. Given $x \in U$, there exists $U_i \ni x$, so define $f(x) = f_i(x)$. This is well defined with respect to the choice of U_i, for if we have $U_j \ni x$, $f_i(x) = \rho_{U_i \cap U_j}^{U_i}(f_i)(x) = \rho_{U_i \cap U_j}^{U_j}(x)$. Because at every x, f can be locally written as f_i, f is meromorphic satisfying the order property necessary for sections in $\mathcal{O}_X[D](U)$. Hence $\mathcal{O}_X[D]$ satisfies the sheaf axiom.

We begin by defining Čech n-cochains over a sheaf of abelian groups. Assume all sheaves hence are sheaves over abelian groups.

Definition 4.3. A **Čech n-cochain** for a sheaf $\mathcal{F}$ over the cover $\mathcal{U}$ is a collection of sections of $\mathcal{F}$, one over each $U_{i_0,\ldots,i_n} = U_{i_0} \cap \cdots \cap U_{i_n}$. The space of Čech n-cochains for $\mathcal{F}$ over $\mathcal{U}$ is denoted by $\check{\mathrm{C}}^n(\mathcal{U}, \mathcal{F})$

$$\check{C}(\mathcal{U}, \mathcal{F}) = \prod_{(i_0,\ldots,i_n)} \mathcal{F}(U_{i_0,\ldots,i_n})$$

Note that $\phi : \mathcal{F} \to \mathcal{G}$ is a sheaf map, there is an induced map on co-chains

$$\phi : \check{C}^n(\mathcal{U}, \mathcal{F}) \to \check{C}^n(\mathcal{U}, \mathcal{G})$$

for any open covering $\mathcal{U}$ sending a cochain $(f_{i_0,\ldots,i_n})$ to $(\phi(f_{i_0,\ldots,i_n}))$

Definition 4.4. The coboundary operator is the map

$$d : \check{C}^n(\mathcal{U}, \mathcal{F}) \to \check{C}^{n+1}(\mathcal{U}, \mathcal{F})$$

which sends n-cochains $(f_{i_0,\ldots,i_n})$ to the $n+1$ co-chain $(g_{i_0,\ldots,i_{n+1}})$ where

$$g_{i_0,\ldots,i_{n+1}} = \sum_{k=0}^{n+1} (-1)^k \rho(f_{i_0,\ldots,\widehat{i_k},\ldots,i_{n+1}})$$

Any n-cochain c satisfying $dc = 0$ is called an n-cocycle, and the group of n-cocycles is called $\check{Z}^n(\mathcal{U}, \mathcal{F})$. Any n cochain which is in the image of the d map is called an n-coboundary, and the n-coboundaries is called $\check{B}(\mathcal{U}, \mathcal{F})$.

After a long calculation, you can check that $d \circ d = 0$.
We have what is called the **Čech cochain complex**

$$0 \to \check{C}^0(\mathcal{U}, \mathcal{F}) \to \check{C}^1(\mathcal{U}, \mathcal{F}) \to \check{C}^2(\mathcal{U}, \mathcal{F}) \to \cdots$$

Definition 4.5. Cohomology with respect to a cover $\mathcal{U}$ is defined using the inclusion $\check{B}^n(\mathcal{U}, \mathcal{F}) \subset \check{Z}^n(\mathcal{U}, \mathcal{F})$. The n-th cohomology group $\check{H}^n(\mathcal{U}, \mathcal{F})$ is defined to be the quotient

$$\check{H}^n(\mathcal{U}, \mathcal{F}) = \check{Z}^n(\mathcal{U}, \mathcal{F}) / \check{B}^n(\mathcal{U}, \mathcal{F})$$

Despite the fact that we just defined cohomology groups above, these are not the Čech cohomology groups. Čech cohomology is defined by attempting to make this cohomology independent of the cover, using something from algebra called a direct limit. We set up the necessary concepts to define the direct limit.

Definition 4.6. Partially order the set of covers by saying $\mathcal{V} \prec \mathcal{U}$, or $\mathcal{V}$ is a refinement of $\mathcal{U}$, if for every open set V_j from $\mathcal{V}$, there exists an open $U_i \in \mathcal{U}$ where $V_j \subset U_i$. This yields a **refinement map** $r : J \to I$ from the index set J of $\mathcal{V}$ to the index set I of $\mathcal{U}$ satisfying $V_j \subset U_{r(j)}$. r induces a map $\check{C}^n(\mathcal{U}, \mathcal{F}) \to \check{C}(\mathcal{V}, \mathcal{F})$

$$\tilde{r}((f_{i_0,\ldots,i_n})) = (g_{j_0,\ldots,j_n})$$

where $g_{j_0,\ldots,j_n} = f_{r(j_0),\ldots,r(j_n)}$

Lemma 4.7. *With the above notation, $\tilde{r}$ induces a map on cohomology groups*

$$H(r) : \check{H}^n(\mathcal{U}, \mathcal{F}) \to \check{H}^n(\mathcal{V}, \mathcal{F})$$

Proof (Exercise IX.3.G): To do this, we show $\tilde{r}$ sends n-coboundaries to n-coboundaries and n-cocycles to n-cocycles. It suffices to show that the d map and $\tilde{r}$ commute. Suppose we have an n-cochain $(f_{i_0,\dots,i_n}) \in \check{C}^n(\mathcal{U},\mathcal{F})$. We have

$$d\tilde{r}((f_{i_0,\dots,i_n})) = d((g_{j_0,\dots,j_n})) = (h_{k_0,\dots,k_{n+1}})$$

where

$$h_{k_0,\dots,k_{n+1}} = \sum_{\ell=0}^{n+1}(-1)^\ell \rho(g_{j_0,\dots,\widehat{j_\ell},\dots,j_{n+1}}) = \sum_{\ell=0}^{n+1}(-1)^\ell \rho(f_{r(j_0),\dots,\widehat{r(j_\ell)},\dots,r(j_{n+1})})$$

But this is the $r_{j_0},\dots,r_{n+1}$ map corresponding to $d(f_{i_0,\dots,i_n})$, so we have shown that the $\tilde{r}$ and d maps commute. $H(r)$ is independent of the refining map r. We check this for refining maps r, r'. Define (ignoring the other h map we just wrote down)

$$h_{\ell_0,\dots,\ell_{n-1}} = \sum_{k=0}^{n-1}(-1)^k f_{r_{\ell_0},\dots,r(\ell_k),r'(\ell_k),\dots,r'(\ell_{n-1})}$$

$$d((h_{\ell_0,\dots,\ell_{n-1}})) = (h'_{\ell_0,\dots,\ell_n})$$

where

$$h'_{\ell_0\dots,\ell_n} = \sum_{i=0}^{n}(-1)^i h_{\ell_0,\dots,\widehat{\ell_i},\dots,\ell_n}$$

$$= \sum_{i=0}^{n}\sum_{k=0,k\neq i}^{n}(-1)^{i+k}\begin{cases} f_{r(\ell_0),\dots,r(\ell_k),r'(\ell_k),\dots,\widehat{r'(\ell_i)},\dots,r'(\ell_n)} & k<i \\ -f_{r(\ell_0),\dots,\widehat{r_{\ell_i}},\dots,r(\ell_k),r'(\ell_k),\dots,r'(\ell_n)} & k>i \end{cases}$$

Grouping terms with common k indices, we get

$$(-1)^{k+1}\Big[(-1)^0 f_{\widehat{r(\ell_0)},\dots,r(\ell_k),r'(\ell_k),\dots,r'(\ell_n)} + \dots + (-1)^{k-1} f_{r(\ell_0),\dots,\widehat{r(\ell_{k-1})},r(\ell_k),r'(\ell_k),\dots,r'(\ell_n)}$$

$$+(-1)^{k+2} f_{r(\ell_0),r(\ell_1),\dots,r(\ell_k),r'(\ell_k),\widehat{r'(\ell_{k+1})},\dots,r'(\ell_n)} + \dots + (-1)^{n+1} f_{r(\ell_0),\dots,r(\ell_k),r'(\ell_k),\dots,\widehat{r'(\ell_n)}}\Big]$$

By the cocycle condition on $(f_{i_0,\dots,i_n})$, we get that the collected terms for the k-th index sum to

$$(-1)^k[(-1)^k f_{r(\ell_0),\dots,\widehat{r(\ell_k)},,r'(\ell_k),\dots,r'(\ell_n)} + (-1)^{k+1} f_{r(\ell_0),\dots,r(\ell_k),\widehat{r'(\ell_k)},\dots,r'(\ell_n)}]$$
$$= f_{r(\ell_0),\dots,\widehat{r(\ell_k)},,r'(\ell_k),\dots,r'(\ell_n)} - f_{r(\ell_0),\dots,r(\ell_k),\widehat{r'(\ell_k)},\dots,r'(\ell_n)}$$

Going back to the sum over the k and i indices, we have

$$h'_{\ell_0\dots,\ell_n} = \sum_{k=0}^{n} f_{r(\ell_0),\dots,\widehat{r(\ell_k)},,r'(\ell_k),\dots,r'(\ell_n)} - f_{r(\ell_0),\dots,r(\ell_k),\widehat{r'(\ell_k)},\dots,r'(\ell_n)}$$

$$= f_{r'(\ell_0),\dots,r'(\ell_n)} - f_{r(\ell_0),\dots,r(\ell_n)}$$

Since the $\tilde{r}((f_{i_0,\dots,i_n}))$ and $\tilde{r}'((f_{i_0,\dots,i_n}))$ sections at the $\ell_0,\dots,\ell_n$ index differ by the section of a coboundary section, we have shown $\tilde{r}'((f_{i_0,\dots,i_n})) = \tilde{r}((f_{i_0,\dots,i_n}))$. ■

We can now denote $H(r) = H^{\mathcal{U}}_{\mathcal{V}} : \check{H}^n(\mathcal{U},\mathcal{F}) \to \check{H}^n(\mathcal{V},\mathcal{F})$ We finally define the Čech cohomology groups:

Definition 4.8. Partially order the set A of covers of X by refinements, and for all covers $\mathcal{U} \in A$, we have the group $\check{H}^n(\mathcal{U}, \mathcal{F})$. Given $\mathcal{V} \prec \mathcal{U}$, we have a map $H^{\mathcal{U}}_{\mathcal{V}} : \check{H}^n(\mathcal{U}, \mathcal{F}) \to \check{H}^n(\mathcal{V}, \mathcal{F})$. We also have for $\mathcal{W} \prec \mathcal{V} \prec \mathcal{U}$, $H^{\mathcal{V}}_{\mathcal{W}} \circ H^{\mathcal{U}}_{\mathcal{V}} = H^{\mathcal{U}}_{\mathcal{W}}$. Finally note that every pair of covers have a common refinement. Now this allows us to define the direct limit over A. Denote

$$\check{H}^n(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U}, \mathcal{F})$$

This is the **Čech Cohomology** of $\mathcal{F}$ on X.

Doing some diagram chasing, we get a **connecting homomorphism** $\Delta : \check{H}^0(X, \mathcal{G}) \cong \mathcal{G}(X) \to \check{H}^1(X, \mathcal{K})$.

Proposition 4.9. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be an onto map of sheaves with kernel $\mathcal{K}$. Then*

$$0 \to \mathcal{K}(X) \xrightarrow{inc} \mathcal{F}(X) \xrightarrow{\phi_X} \mathcal{G}(X) \xrightarrow{\Delta} \check{H}^1(X, \mathcal{K}) \xrightarrow{inc_*} \check{H}^1(X, \mathcal{F}) \xrightarrow{\phi_*} \check{H}^1(X, \mathcal{G})$$

is an exact sequence.

Here is why this exact sequence can be useful. Suppose you know just one of the groups in the exact sequence—say $\check{H}^1(X, \mathcal{K})$. You can then make conclusions about groups or maps in between, such as that ϕ_X is surjective if $\check{H}^1(X, \mathcal{K}) = 0$. If we write the exact sequence of sheaves

$$0 \to \mathcal{K} \to \mathcal{F} \xrightarrow{\phi} \mathcal{G} \to 0$$

we conclude by the first isomorphism theorem that $\mathcal{G}(X)/\phi_X(\mathcal{F}(X)) \cong \check{H}^1(X, \mathcal{K})$ since $\phi_X(\mathcal{F}(X)) = \ker(\Delta)$ and $\mathrm{im}(\Delta) = ker(inc_*) = \check{H}^1(X, \mathcal{K})$. We calculate the Čech homology groups in the case of the sheaf of $\mathcal{C}^\infty$ functions. In the following example, we use a partition of unity on a Riemann surface X.

Proposition 4.10. *Let X be a Riemann surface. Then given an open covering $\mathcal{U} = \{U_i\}$ there exists a set of $\mathcal{C}^\infty$ functions $\{\varphi_i\}$ such that*

- *for every point p, there exists a neighborhood of p intersecting finitely many of the supports of φ_j*
- *for all points p, $\sum_i \varphi_i(p) = 1$*
- *the support of φ_i is contained in U_i for all i*

Example 4.11. We show that the cohomology groups $\check{H}^n(X, \mathcal{U})$ are zero for $n \geq 1$ and open coverings $\mathcal{U}$ of X. This would then imply the direct limit $\check{H}^n(X, \mathcal{C}^\infty) = 0$ for all $n \geq 1$. We do this for the case when $n = 1$; the rest of the cases involve a similar calculation but with more indices.
Fix an open covering $\mathcal{U}$, and let (f_{ij}) be a 1-cocycle for the sheaf $\mathcal{C}^\infty$ on $\mathcal{U}$. Consider the $\mathcal{C}^\infty$ function $\varphi_j f_{ij}$ and extend it by zero outside $Supp(\varphi_j)$. This is smooth because it is locally. We can then define this function on U_i. Now define

$$g_i = -\sum_j \varphi_j f_{ij}$$

Because each of the terms in the sum are $\mathcal{C}^\infty$, g_i is $\mathcal{C}^\infty$. Now we have

$$g_j - g_i = -\sum_k \varphi_k f_{jk} + \sum_k \varphi_k f_{ik} = \sum_k \varphi_k (f_{ik} - f_{jk}) = \sum_k \varphi_k f_{ij} = f_{ij}$$

where the second to last equality follows from the fact that f_{ij} is a 1-cocycle. Hence $(f_{ij}) = d(g_i)$ is a coboundary. By a similar argument, the n-th cohomology group for the sheaf $\mathcal{E}^{(0,1)}$ of $(0,1)$ forms also vanishes for $n \geq 1$. The book applies this to calculate the cohomology groups for the sheaf $\mathcal{O}$ of holomorphic functions in the following manner: the short exact sequence (the onto condition of the second to last map is by Dolbeault's lemma)

$$0 \to \mathcal{O} \to \mathcal{C}^\infty \xrightarrow{\overline{\partial}} \mathcal{E}^{0,1} \to 0$$

and the resulting exact sequence of cohomology groups

$$\check{H}^n(\mathcal{E}^{0,1}) \xrightarrow{\Delta} \check{H}^{n+1}(\mathcal{O}) \to \check{H}^{n+1}(\mathcal{C}^\infty)$$

can be used to conclude that $\check{H}^n(\mathcal{O}) = 0$ for all $n \geq 2$.

Example 4.12. We also show that cohomology groups for skyscraper sheaves vanish for dimension greater or equal to 1. We first generalize the definition of skyscraper sheaves to make the calculation appear more natural. Previously, we defined skyscraper sheaves as the pushforward of a constant sheaf by $i_p : \{p\} \to X$ where p is a single point. We define a skyscraper sheaf to be a totally discontinuous sheaf on which sections have discrete support. In other words, we associate a group G_p for each $p \in X$ and sections on an open set U are elements of

$$\prod_{p \in U} G_p$$

where the coordinates at all but a discrete set of points vanish. Note that the previous example of a skyscraper sheaf is an example of this one. Now let X be a space and $\mathcal{F}$ be a skyscraper sheaf on X. We would like to show $n \geq 1$ implies $\check{H}^n(X, \mathcal{F}) = 0$. It suffices to prove that the cohomology groups corresponding to each cover $\mathcal{U} = \{U_i\}$ vanish. We also apply a partition of unity argument for this proof. Totally order the index set of $\mathcal{U}$. Defining

$$\varphi_i(p) = \begin{cases} 1 & p \in U_i - \cup_{j<i} U_j \\ 0 & otherwise \end{cases}$$

This satisfies the conditions for a partition of unity on X, only except that the functions are integer valued and mostly discontinuous. Note that if $f \in \mathcal{F}(U)$ is a section, $\varphi_j f$ is also a section in U. For all but finitely many points, $\varphi_j(p) \cdot f(p) = \varphi_j(p) \cdot 0 = 0$. Hence $\varphi_j \cdot f$ is a section on U. Let (f_{ij}) be a 1-cocycle for $\mathcal{F}$ on $\mathcal{U}$. Consider $\varphi_j f_{ij}$ and extend by zero outside of the support of φ_j. This extension can then be restricted to U_i. Let $g_i = -\sum_j \varphi_j f_{ij}$, which is a section on U_i. As before, $d(g_i) = (f_{ij})$, so all 1-cocycles of $\mathcal{F}$ on $\mathcal{U}$ are coboundaries.

Example 4.13. As a final short remark, we can connect sheaf theory to algebraic topology in the following sense. The Čech cohomology groups of the locally constant sheaf are precisely the simplicial cohomology groups for triangulable spaces. A proof of this fact can be found in Munkre's book on Algebraic Topology [Mun93].

5 Conclusions

Sheaf theory is quite deep, and it has connections to many fields. As seen in the exposition above, we made connections to algebra, complex analysis, and topology. We directly associated algebra to our sheaves by looking at sheaves over groups, and in particular abelian groups for Čech cohomology. This yielded cohomology groups, which we could solve for algebraically in exact sequences. In our examples we examined sheaves on Riemann surfaces, which connect strongly to complex analysis. We also noted that on triangulable spaces Čech cohomology groups of the locally constant sheaf are the simplicial cohomology groups from algebraic topology. Our exposition merely scratches the surface of sheaf theory.

References

[Mir95] Miranda, R. *Algebraic Curves and Riemann Surfaces.* American Mathematical Society, 1995.

[Mun93] Munkres, J. *Elements of Algebraic Topology.* CRC press, 1st edition, 1993.

[Vak15] Vakil, R. *The Rising Sea, Foundations of Algebraic Geometry.* 2015.