5320 Final Project: Math & Music Theory

Hank Jennings

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1 Introduction

Music and math have always shared a deep rooted connection with each other. In many ways, math is the hidden language behind music, describing how we hear certain sounds in relation to another and even how those sounds are interpreted by our brains in the first place. For example, consider the twelve pitch-classes generated by the twelve distinct pitches present in Western Music Theory, in which each pitch is assigned a letter from A to G (with possible sharp and flat accidentals), based on the frequency of the respective sound-wave in Hz. Notes with twice the frequency as another note are considered to be in the same pitch-class and thus identified with the same letter symbol. In this way, a sound-wave with a frequency of 440 Hz would be labeled A, and any halving or doubling of that frequency value such as 220 Hz or 880 Hz would also result in an A pitch, being one octave below or above the original, respectively. This is precisely because our ears hear sounds on a base 2 logarithmic scale!

Furthermore, using equal tempered tuning, the difference between any two pitches that are n semitones apart expressed as a ratio is exactly equal to $\sqrt[12]{2}^n$. The value of $\sqrt[12]{2}$ is chosen because of its innate tendency of it's powers to be within 0.1% accuracy of relatively simple rational numbers (i.e. those with small numerators and denominators). For example, $\sqrt[12]{2}^5 \approx 1.3348 \approx 4/3$, $\sqrt[12]{2}^7 \approx 1.4983 \approx 3/2$. The resulting consonance or dissonance of the two pitches played against each other can be expressed as a ratio between the two frequencies, where those with smaller integer values can be seen as more harmonious than those with relatively larger values. In this way, two notes that are seven semitones apart will have a frequency ratio of $\sqrt[12]{2}^7 \approx 3/2$ which will sound very harmonious (and is often called a *perfect fifth*) while two notes that are six semitones apart will result in a frequency ratio of $\sqrt[12]{2}^6 \approx 45/32$ which sounds very dissonant (and is often referred to as a *tritone*). When any two distinct pitches are played together, our brain picks up on the polyrhythms created by the two frequencies of sound-waves expressed by these ratios. Naturally, those polyrhythms that are easier to interpret corresponding to lower frequency ratios will sound better than those with high ratios. The consonance or dissonance of two different pitches can alternatively be described with the harmonic overtone series, which details the natural phenomenon of multiple overtones being created and heard at the same time any single pitch is made. Each of those overtones are equal to 1/nth of the frequency of the original pitch for n going to infinity, with the volume of each overtone generally decreasing as n increases. In this way, pitches that share one or more overtones sound more harmonious, and it is easy to see that pitches which are relatively small ratio frequencies of other pitches will naturally share more audible overtones. In fact, this is exactly why the harmonic series represented by $\sum_{n=1}^{\infty} 1/n$ is called the harmonic series! Again, the twelfth root of two is particularly powerful as a generator for the given pitches of a pitch class due to its close accuracy to the resulting ratio frequencies created by the harmonic overtone series.

This is all very interesting, but the math we've been exploring has been pretty simple thus far. In order to identify some of the more fundamental ideas of music theory from a mathematical perspective, we must dive into the world of group theory. Our earlier look at the twelve distinct pitch classes created through equal tempered tuning proves quite useful in this regard, as we can now view these twelve pitch classes as isomorphic to the set $\mathbb{Z}/12$. Setting the pitch C be equal to zero and each resulting pitch to be equal to the amount of half-steps (or semitones) needed to traverse to that pitch going up starting from C, we can view movement up and down the chromatic scale as addition and subtraction by half-steps to the respective note modulo 12. In this way, we can look at the various changes of note or chord shapes a chord can take as actions on this set and gain insight to some powerful math at work behind the scenes. First, however, we must make some formal definitions.

2 Group Theory

Definition For G a group with identity e and X a set, a *(left) group action* is a function $G \times X \to X$, $(g, x) \mapsto g.x$ such that

- $e.x = x$
- $g.h.x = (gh).x$

A right group action is defined similarly as a function $X \times G \to X$, $(x, g) \mapsto x.g$ such that

- $x.e = x$
- $x.g.h = x.(gh)$

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Definition A *transitive* group action is a group action with the additional property that for every pair of elements $x, y \in X$, there exists $g \in G$ such that $g.x = y$.

Definition A *faithful* group action is a group action in which for every set of maps $gx = hx$, $g, h \in G$, $g = h$.

Definition A *simply transitive* group action is a group action that is both transitive and faithful.

Definition The *orbit* of a group action acting on an element $s \in S$ is the set of all $g.s$ such that $g \in G$. In this way, it is clear to see that any transitive group action has only one orbit which consists of all the elements $s \in S$, since any arbitrary element $s \in S$ can be sent to any other element $s' \in S$

Definition A left action of G on itself is a group action $G \times G \to G$, $(g, x) \mapsto g.x$. Of particular note, this group action is always simply transitive.

Definition A right action of G on itself is a group action $G \times G \to G$, $(x, g) \mapsto x.g$. As before, these group actions are also always simply transitive.

Definition The *centralizer* of a subgroup H of a group G is the set of all group actions $g \in G$ such that $gh = hg \ \forall h \in H$.

Definition Two groups are called *dual* if they are both centralizers of the other.

One important theorem from group theory that allows us to relate any group G to a subset of the Symmetric Group $Sym(G)$ is Cayley's Theorem, which is stated and proved below.

Theorem (Cayley's Theorem) For any group G, there is an isomorphism between G and some subset $H \subseteq Sym(G)$. I.e., $G \cong H$.

Proof. First consider the left group action of G on itself, $G \times G \to G$, $(g, x) \mapsto g.x$. We will show that this correspondence is bijective. Let $g, g' \in G$ such that $(g, x) = (g', x) \iff g.x = g'.x \iff g = g'$. This shows that the group action is injective. Now consider the function $(g, (g^{-1}, x)) = g.g^{-1}.x = x$, so the group action has a preimage and is thus surjective. This shows that any group action on itself is bijective, or in other words it is a permutation on the set underlying the group S_G. We can now consider the mapping $\phi : G \to Sym(G)$ defined by that permutation representation. By examining the kernel of ϕ, we see that $\forall g \in ker\phi$, $g.e = ge = g = e$, i.e. $ker\phi$ is trivial so ϕ is injective. This is equivalent to saying that ϕ is faithful. We now note that $Im\phi = H \subseteq Sym(G)$, and our final result comes from the First Isomorphism Theorem, which tells us that $G/ker\phi \cong Im\phi$, or equivalently, $G \cong H$. $\square$

Though not as well known and less broad in scope, there is one other theorem of Cayley's that will have important significance in our discussion of music and mathematics. The theorem will be stated here, but the proof will be held until later in our discussion.

Theorem (Cayley's Dual Theorem) If G is a group, then we obtain dual groups via the two embeddings of G into $Sym(G)$ as left and right actions of G onto itself. All dual groups arise in this way.

Okay, so we've got some real formal mathematical ideas here now, but these definitions are all pretty abstract. How does this all tie into music? It is hear that our discussion of the 12 pitch classes as equivalent to $\mathbb{Z}/12$ becomes truly useful. Viewing each of the pitch class integers as twelve equidistant points on a circle (similar to a clock's hour figures), we can see the shape that results from connecting these points outer perimeter is a dodecagon. Then, by examining the 24 possible symmetries of that dodecagon, which include the 12 rotations from one point to a proceeding point and the 12 lines of symmetry we can flip the dodecagon across, we get the dihedral group D_{12} of order 24! Great, so we've got a group, but how does it act on itself? And what do these group actions have to do with music? Once again we must dive into some more definitions, but this time we get to look not only through a mathematical lens but also through a musical one as we tie in some ideas from western music theory.

3 Major and Minor Triads and the T/I group

Definition A *major third* is a musical term used to express two pitches that are four semitones apart. A *minor third* describes two pitches three semitones apart.

Definition A *major triad* is the stacking of one minor third on top of a major third using only three notes. Similarly, a *minor triad* is a major third stacked on top of a minor third.

Example Represented mathematically, a C-major triad (commonly denoted by an upper case C) would consist of the root note C, the major third note E four semitones above the root, and the minor third note G three semitones above the major third E. In our translation to the integers modulo 12, this would correspond to the set of integers {0,4,7}. One example of a minor triad can be seen in the f-minor triad (minor triads typically have lower case pitch classes as symbols) which is constructed from the root F, the minor third note A♭ three semitones above the root, and the major third note C four semitones above the minor third A♭. Our corresponding representation in the integers modulo 12 defined by our mapping earlier would give {5,8,0}.

Definition A *transposition* is a function $T : \mathbb{Z}/12 \to \mathbb{Z}/12$ defined by $T_n(x) = x + n \bmod 12$. Transpositions of a major or minor triad involve the movement of each of it's individual notes up or down a set amount of semitones. Transposed major triads will always be major, and transposed minor triads will always be minor.

Example The transposition T_2 on the C-major interval involves transposing each note up two semitones. The resulting triad is a D-major triad, and is represented as a set in $\mathbb{Z}/12$ as $\{2, 6, 9\} = \{D, F\sharp, A\}$. Notably, A transposition function T_n can be seen as a rotation of the dodecagon by $\frac{n}{12}$ of a turn.

Definition An *inversion* is a function $I : \mathbb{Z}/12 \to \mathbb{Z}/12$ defined by $I_n(X) = -x + n$ mod 12. An inversion I_n moves a major triad to the minor triad $5 + n$ mod 12 semitones above the root, and moves a minor triad to the major triad $7 + n$ mod 12 semitones above the root (or $5 - n$ mod 12 semitones below the root, since $7 = -5$ mod 12). Inversions of major triads will always be minor and inverted minor triads will always be major.

Remark It should be noted that the inversion function described here is distinctly different from the of an inversion in the typical music theory sense, which involves the rearrangement of a chord's notes to have a different root note defined by the degree of the inversion.

Example The inversion I_0 of a C-major triad involves flipping the coordinates about the 0 - 6 axis. The resulting triad will have its values {0,4,7} mapped to $\{0, 8, 5\} = \{C, A\sharp, F\} = \{F, A\sharp, C\}$, making it an f-minor triad. The same inversion on that f-minor triad would result back in the original C-major triad. In this sense, an inversion function can be seen as a rotation of the dodecagon by $\frac{n}{12}$ of a turn followed by a flip over the resulting vertical axis.

The importance of these transposition and inversion functions becomes apparent when we consider the set of all of them as a group T/I. Of note, there are 24 elements in this group, since there are 12 transpositions and 12 inversions. Furthermore, since $I_n = T_n \circ I_o$, and considering that T_n has order 12 and I_0 has order 2, we see that T/I is generated by $\{T_1, I_0\}$. What's more, through the use of it functions, it acts (from the left) on the group S of all major and minor triads, which is also of order 24 (12 major triads and 12 minor triads). Hmm, interesting ... but that's not all! If we consider the fact that T_n can send any major or minor triad to another respective major or minor triad uniquely, and that I_n can similarly send any major triad uniquely to a minor triad or vice versa, we notice that the T/I group acting on the group S is *simply transitive*! This is very interesting! We'll come back to this later.

4 The PLR Group

Definition A function $P : S \to S$ defined by $P(y_1, y_2, y_3) = I_{y_1+y_3}(y_1, y_2, y_3)$ is called a *parallel* function. This function maps a major triad to the corresponding minor triad of the same pitch class and vice versa. For example, $P(0,4,7) = (7,3,0) \iff P\{C,E,G\} = \{C,E\flat,G\} \iff P(C) = c$.

Definition A function $L : S \to S$ defined by $L(y_1,y_2,y_3) = I_{y_2+y_3}(y_1,y_2,y_3)$ is called a *leading voice exchange* function. This function maps a major triad to the minor triad four semitones above the root of the major triad and sends a minor triad to the major triad four semitones below the root. For example, $L(0,4,7) = (11,7,4) \iff L\{C,E,G\} = \{E,G,B\} \iff L(C) = e$.

Definition A function $R : S \to S$ defined by $R(y_1,y_2,y_3) = I_{y_1+y_2}(y_1,y_2,y_3)$ is called a *relative* function. This function maps a major triad to the minor triad three semitones below the root of the major triad and sends a minor triad to the major triad three semitones above the root. For example, $L(0,4,7) = (9,0,4) \iff L\{C,E,G\} = \{A,C,E\} \iff L(C) = a$.

Remark Looking at these functions geometrically, the axis in which the pitches are flipped over is not always the vertical axis, as was the case in our earlier defined inversion function. Instead, each function P, L, and R have a unique axis of rotation, being the lines spanned by $\frac{y_1+y_3}{2}$ to $\frac{y_1+y_3}{2}+6$, $\frac{y_2+y_3}{2}$ to $\frac{y_2+y_3}{2}+6$, and $\frac{y_1+y_2}{2}$ to $\frac{y_1+y_2}{2}+6$, respectively.

By examining these functions more closely, we can make a couple important distinctions about the PLR group. Firstly, we note that we can generate the function P using the functions L and R. We see this by example with the major C chord. By alternatively applying R and L, we see that any starting major triad will be mapped to the minor triad three semitones below the root, which will then be mapped to the major triad four semitones below that root. This corresponds to moving a major triad seven semitones below the root in total for each iteration of R and L used. With this in mind, applying R and L three times each in succession would map a C major triad to the major triad $7 * 3 = 21 \equiv 9 \mod 12$ below it, corresponding to $E\flat$ major. This can be seen as $(LR)^3(C) = E\flat$. By applying the R function one more time, we get the relative minor of $E\flat$, which is another three notes below the root, resulting in c minor! Thus the PLR group is generated by L and R alone. Looking at this relationship further, we can see that (LR) has order 12 (since it maps major triads to major triads seven semitones below the root and maps minor triads to minor triads seven semitones above the root and since 7 and 12 are coprime). Thus, by routinely alternating between L and R for any starting given triad, we generate the whole group of major and minor triads of order 24! Even better, through this process, we have a unique mapping from any one major or minor triad to any other major or minor triad, so the PLR group is simply transitive, acting on the dihedral group of order 24 on the right!

5 Dual Groups

We now return to the earlier theorem of Cayley's, which states that dual groups are formed by a group acting on itself from the left and the right. The proof follows below.

Proof. Let G_1 and H_1 be the two groups formed by an arbitrary group G acting on itself from the left and the right. Let $g \in G_1$ and $h \in H_1$ be two group actions acting on an element $x \in G$. Now suppose G_1 and H_1 are not dual, i.e., $g.(x.h) \neq (g.x).h \iff gh^{-1}x \neq h^{-1}gx$. Multiplying each side by g^{-1}, we get $g^{-1}gh^{-1}x \neq g^{-1}h^{-1}gx \iff x.h \neq g.x.hg \iff xh \neq xh$ which leads to a contradiction. Thus G_1 and H_1 are dual. □

Using this framework, we can now see that the T/I group and the PLR group are dual since they are both of order 24,simply transitive, and act on the group G from the left and right. In the context of these groups, this means that if you apply a transposition/inversion along with any element from the PLR group, it would not matter what order you applied the funtions in as they would result in the same ending chord. That's actually really cool! It was very interesting learning about all the in-depth math behind musical concepts such as chord changes and melodic motion. Math really is the universal language of the universe!

6 Bibliography

Crans, Allisa S, et al. "Musical Actions of Dihedral Groups." Crans, 2011, www.maa.org/sites/default/files/pdf/upload$_library/22/Hasse/Crans2011.pdf$.